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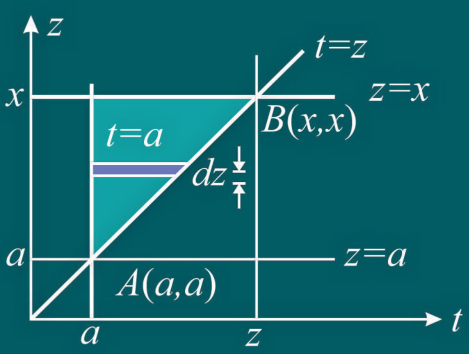
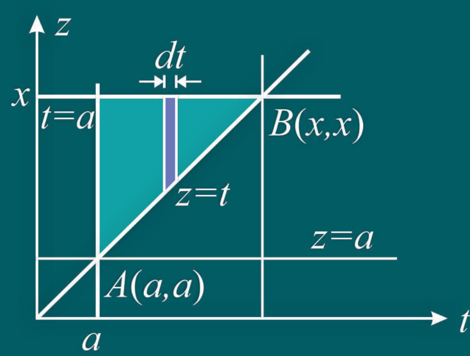
INTEGRAL EQUATIONS





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Preface

Integral Equations are being used as an essential effective tool by the mathematicians, engineers and theoretical physicists to understand and solve research problems in their field. Two-point initial value problems and boundary value problems with fixed and variable boundary are often encountered by researchers.

Integral equations have now been established as a far effective and highly useful branch in almost all disciplines of knowledge. Hence, this course has been taken as an indispensable part of syllabi of all Indian universities at postgraduate level. Not only that, in competitive examinations like NET/SET, this course inherits importance. Therefore, it becomes necessary for students and teacher alike to follow the concepts and operations of integral equations.

Looking to these requirements, we have tried to provide the concepts and principles quite clearly and in an organised manner. In order to make the book user-friendly, we have presented the matter in a simple way and it is complete in all respects from the examination point-of-view. By presenting sufficient number of assorted examples, we have tried to inculcate the habit and create confidence in students to try to solve more and more problems on their own.

Because of author's long experience of teaching and also being actively engaged in research, it is expected that the book shall prove immensely beneficial to the students for whom it is meant.

All suggestions for the improvement of the book shall be thankfully acknowledged.

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M.C. Goyal

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We are also thankful to our family members for taking all the care during long hours at the desk.

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CHAPTER 1

Basic Concepts

1.1 INTRODUCTION

Various physical problems in physics and other applied fileds culminate into initial value problems or boundary value problems. Although it is equivalent to frame the problem in the form of (ordinary and partial) differential equations or in the form of integral equations, but it is preferred to choose the integral form due to two main reasons. Firstly, the solution of integral equation is much easier than the original boundary value or the initial value problems. The second reason lies in the fact that integral equations are better suited to approximate methods than differential equations. Moreover, integral equations develop as representation formulae for the solution of differential equations. We will find in the forthcoming chapters that differential equations can be replaced by an integral equation with the help of initial and boundary conditions. As a result, each solution of the integral equation satisfies the boundary conditions itself.

1.2 ABEL'S PROBLEM

In 1826, Abel obtained an integral equation by considering the motion of a material point $P(x, y)$ under the action of force of gravity moving in vertical plane (ξ, η) along some smooth curve. It is required to establish the curve such that the material point P, starting from rest at $P(x, y)$ reaches the point $Q(\xi, \eta)$ at any instant t. Let T be the time taken by the particle from P to the lowest point O, the origin of coordinates and axes, as shown in Fig. 1.1. Let $\overset{\frown}{OQ} = s,$ then the velocity of particle at Q is

$$\frac{ds}{dt} = -\sqrt{2g(x-\xi)}$$

$$\therefore \qquad [t]_0^T = -\int_P^Q \frac{ds}{\sqrt{2g(x-\xi)}}$$

$$\therefore \qquad T = \int_O^P \frac{ds}{\sqrt{2g(x-\xi)}}$$

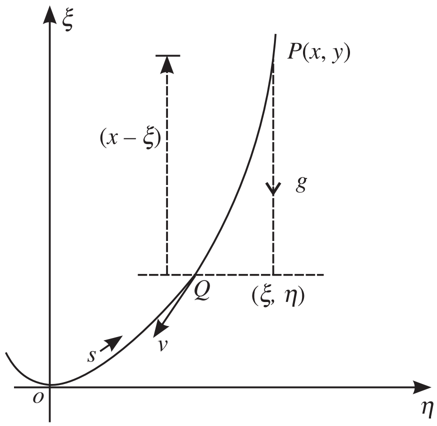

Fig. 1.1 Particle falling under gravity along the curve PQO.

Now, if the shape of curve is given, then *s,* and hence, *ds* can be expressed in terms of ξ. We take

$$ds = u(\xi)d\xi, \text{ and then}$$

$$T = \int_0^x \frac{u(\xi)\,d\xi}{\sqrt{2g(x-\xi)}}$$

Here, if the curve is such that the time of descent T is a function of x, say, $f(x)$, then the above relation shapes as

$$f(x) = \int_0^x \frac{1}{\sqrt{2g(x-\xi)}} u(\xi)\,d\xi$$

This leads to find the unknown function $u(\xi)$ and we get Abel's integral equation.

During the analysis, it is found that an initial value problem is always converted to a Volterra integral equation, while a boundary value problem is always converted to a Fredholm integral equation. Before we take the classification of integral equations, it is worth to be familiar with the initial value problems and the boundary value problems.

1.3 INITIAL VALUE PROBLEM AND BOUNDARY VALUE PROBLEM

Initial value problem

When an ordinary differential equation is solved under conditions which involve dependent variable and its derivative at the same value of its independent

variable, then the problem under consideration is said to be an *initial value problem*.

For instance

$$y''' + xy'' + 2(x^2 - x)y = e^x - x,$$

with the conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = k$

Boundary value problem

When an ordinary differential equation is solved under conditions which involve dependent variable and its derivative at two different values of independent variable, then the problem under consideration is said to be a *boundary value problem*.

For example

$$y'' + xy' + 2y = e^{-x}$$

with the conditions $y(0) = 0$, $y'(1) = -1$

1.4 INTEGRAL EQUATION

An *integral equation* is an equation in which one unknown function (which is to be determined) appears under one or more integral signs. If in the equation, the derivatives of this unknown function are also present, it is called an *integro-differential equation*.

For example, for $a \le x \le b$, $a \le \xi \le b$, the equations

(linear) $$u(x) = f(x) + \lambda \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi, \tag{1.1(a)}$$

(linear) $$u(x) = \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi \tag{1.1(b)}$$

(non linear) $$u(x) = \lambda \int_a^b K(x, \xi) \cdot [u(\xi)]^3\, d\xi \tag{1.1(c)}$$

are integral equations. $u(x)$ is an unknown function, $f(x)$, $K(x, \xi)$ are known functions and λ, a, b are constants. The functions involved may be complex valued functions of real variables x and ξ.

$$\frac{du}{dx} + \lambda \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi = f(x),\ a \le x \le b \tag{1.1(d)}$$

is integro-differential equation of unknown function $u(x)$.

The known bivariate function $K(x, \xi)$ which is integrable in the domain $a \le x \le b$, $a \le \xi \le b$ is called the *kernel* of the equation; the known function $f(x)$ is a continuous function. In the book, we shall find various forms of this kernel and will establish the solution of the integral equation consequently.

An integral equation is called *linear* if only linear operations are performed in it upon the unknown functions, which means it is the equation in which no non-linear functions of the unknown function are involved. Equations [1.1(a)] and [1.1(b)] are linear equations. An equation which is not linear is known as *non-linear* integral equation, as shown in Eq. [1.1(c)].

The most general linear integral equation may be shaped as

$$v(x) \cdot u(x) = f(x) + \lambda \int_a K(x, \xi) \cdot u(\xi)\, d\xi \tag{1.2}$$

where the upper limit of the integral may be variable or constant. The functions v, f, K are known functions and u is to be determined, λ is a non-zero parameter, which may be real or complex.

The importance of keeping λ separate from K lies in the fact that it plays an essential role in the theoretical arguments for the problem under context.

Kinds of linear integral equations

1. First kind, if $v = 0$, Eq. (1.2) reduces to

$$f(x) + \lambda \int_a K(x, \xi) \cdot u(\xi)\, d\xi = 0$$

2. Second kind, if $v = 1$, Eq. (1.2) provides

$$u(x) = f(x) + \lambda \int_a K(x, \xi) \cdot u(\xi)\, d\xi$$

3. Third kind, if $v \neq 0$, Eq. (1.2) itself works.

1.5 SPECIAL KINDS OF KERNELS

As mentioned earlier, the role of the known bivariate function $K(x, \xi)$ is quite significant both from the problem and its solution point-of-view. Mainly, we shall come across with the following forms:

1. Symmetric kernel: The kernel $K(x, \xi)$ is symmetric (complex symmetric is also called Hermitian) if

$$K(x, \xi) = \bar{K}(\xi, x)$$

where the bar represents the complex conjugate. A real kernel is symmetric if $K(x, \xi) = K(\xi, x)$. For instance $e^{\xi x}$ is symmetric, while $\tan^{-1}\left(\frac{\xi}{x}\right)$ is not a symmetric kernel.

2. Separable or degenerate kernel: If $K(x, \xi) = \sum_{i=1}^{n} g_i(x).\, h_i(\xi)$ means K has been expressed as the sum of a finite number of terms, each of which is the product of function of x only and ξ only, then such a kernel is called *separable or degenerate kernel*. Obviously, $g_i(x)$ and $h_i(\xi)$ are linearly independent (or else some of the terms will combine, and consequently, the number of terms will reduce). A degenerate kernel has a finite number of characteristic values.

3. Difference kernel: A kernel of the form $K(x - \xi)$ is called *difference kernel*.

1.6 CLASSIFICATION OF INTEGRAL EQUATION

Integral equations are classified into the following four classes:

1. Fredholm integral equation
2. Volterra integral equation
3. Singular integral equation
4. Convolution integral equation

Fredholm integral equation

A linear integral equation of the form

$$v(x) \cdot u(x) = f(x) + \lambda \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi \tag{1.3}$$

where a and b both are constants is called *Fredholm integral equation of the third kind*. Here, $f(x)$, $v(x)$ and $K(x, \xi)$ are known functions, $u(x)$ is unknown function and λ is real or complex parameter.

Now, if in Eq. (1.3), we set $v(x) = 0$, then we get

$$f(x) + \lambda \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi = 0 \tag{1.4}$$

and it is *Fredholm integral equation of the first kind*.

Next, if in Eq. (1.3), $v(x) = 1$, we get

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi \tag{1.5}$$

and it is *Fredholm integral equation of the second kind*. If in Eq. (1.5), $f(x) = 0$, i.e.,

$$u(x) = \lambda \int_a^b K(x, \xi) \cdot u(\xi)\, d\xi \tag{1.6}$$

it is known as *homogeneous Fredholm integral equation of the second kind*.

Volterra integral equation

A linear integral equation of the form

$$v(x) \cdot u(x) = f(x) + \lambda \int_a^x K(x, \xi) \cdot u(\xi)\, d\xi \tag{1.7}$$

where the upper limit of the integral is variable, $v(x)$, $f(x)$ and $K(x, \xi)$ are known functions and $u(x)$ is unknown function, is said to be *Volterra integral equation of the third kind*. As usual λ is a real or complex parameter and the function $K(x, \xi)$ is the kernel of the integral equation.

If we set $v(x) = 0$, i.e., Eq. (1.7) takes the form

$$f(x) + \lambda \int_a^x K(x, \xi) \cdot u(\xi)\, d\xi = 0 \tag{1.8}$$

then it is called *Volterra integral equation of the first kind*.

Again, if $u(v) = 1$, Eq. (1.7) takes the shape

$$u(x) = f(x) + \lambda \int_a^x K(x,\xi) \,.\, u(\xi) d\xi \tag{1.9}$$

then it is called *Volterra integral equation of the second kind.*

Further, if in Eq. (1.9), $f(x) = 0$, then it becomes

$$u(x) = \lambda \int_a^x K(x,\xi) \,.\, u(\xi) d\xi \tag{1.10}$$

and is called *homogeneous Volterra integral equation of the second kind.*

Singular integral equation

An integral equation is a *singular integral equation*, if either

1. One or both the limits of integration are infinite, or

2. The kernal becomes infinite at one or more points within the range of integration. For instance,

$$u(x) = f(x) + \lambda \int_{\infty}^{-\infty} e^{-|x-\xi|} u(\xi) d\xi$$

and
$$u(x) = \int_0^x \frac{1}{(x-\xi)^\alpha} \cdot u(\xi) d\xi, \quad 0 < \alpha < 1$$

are both singular integral equations.

Convolution integral equations

If the kernel $K(x, \xi)$ is a function of one variable and is of the type $K(x, \xi) = K(x - \xi)$, then the integral equations, i.e., Eqs. (1.5) and (1.9) take the shape

$$u(x) = f(x) + \lambda \int_a^b K(x-\xi) \cdot u(\xi) d\xi \tag{1.11}$$

and
$$u(x) = f(x) + \lambda \int_a^x K(x-\xi) \cdot u(\xi) d\xi \tag{1.12}$$

respectively, and are called *integral equations of convolution type*.

Further, if $u_1(x)$ and $u_2(x)$ are two continuous functions defined for $x \geq 0$, then the convolution or faltung of u_1 and u_2 is expressed or defined by

$$u_1 * u_2 = \int_o^x u_1(x-\xi) \cdot u_2(\xi) d\xi = \int_o^x u_1(\xi) \cdot u_2(x-\xi) d\xi \tag{1.13}$$

1.7 ITERATED KERNELS

1. For the Fredholm integral equation of the second kind [Eq. (1.5)], we define the iterated kernel, $K_n(x, \xi)$, $n = 1, 2, 3, \ldots$ as below:

$$K_1(x, \xi) = K(x, \xi)$$

and
$$K_n(x,\xi) = \int_a^b K(x,z) K_{n-1}(z,\xi) dz, \ n = 2,3 \tag{1.14}$$

2. For the Volterra integral equation of the second kind [Eq. (1.9)], we define the iterated kernels $K_n(x, \xi)$, $n = 1, 2, 3, \ldots$ as below:

$$K_1(x, \xi) = K(x, \xi)$$

and $$K_n(x,\xi) = \int_\xi^x K(x,z)K_{n-1}(z,\xi)dz, \quad n = 2,3,\ldots \tag{1.15}$$

1.8 RECIPROCAL KERNAL OR RESOLVENT KERNEL

Let the solutions of integral equations, i.e., Eqs. (1.5) and (1.9) of the second kind be:

$$u(x) = f(x) + \lambda\int_a^b R(x,\xi;\lambda)f(\xi)d\xi \tag{1.16}$$

and $$u(x) = f(x) + \lambda\int_a^x S(x,\xi;\lambda)f(\xi)d\xi \tag{1.17}$$

respectively, then $R(x, \xi; \lambda)$ and $S(x, \xi; \lambda)$ are called the *reciprocal or resolvent kernel* of the respective integral equation.

1.9 EIGENVALUES AND EIGENFUNCTIONS

We consider the homogeneous Fredholm integral equation

$$u(x) = \lambda\int_a^b K(x,\xi)u(\xi)d\xi \tag{1.18}$$

for which $u(x) = 0$ is an obvious solution. This solution is taken as *zero or trivial solution*.

The values of the parameter λ for which Eq. (1.18) possesses non-zero solutions $[u(x) \neq 0]$ are the *eigenvalues* of Eq. (1.18) or of the kernal $K(x, \xi)$. Further, corresponding to such eigenvalues of λ, every non-zero solution of Eq. (1.18) is called an *eigenfunction*.

Remarks: 1. Eigenvalues are also termed as *characteristic values* or *characteric numbers*. Similarly, eigenfunctions are also called *characteristic functions* or *fundamental functions*.

2. $\lambda = 0$ is not considered as eigenvalue, since it corresponds to $u(x) = 0$.
3. Corresponding to eigenvalue λ, $u(x)$ and $C\,u(x)$ are eigenfunctions for arbitrary constant C.
4. The set of all characteristic numbers of an integral equation with a kernel $K(x, \xi)$ is called the *spectrum* of the kernel (or of the integral equation).
5. A homogeneous Fredholm integral equation may not possess any real eigenvalue or eigenfunction.

1.10 SOLUTION OF AN INTEGRAL EQUATION

A solution of an integral equation [such as Eqs. (1.3) and (1.7)] is a function $u(x)$, which when substituted in the equation, reduces it to an identity. For example, for the integral equation.

$$u(x) = 1 + \int_0^x u(\xi)d\xi$$

the solution is $u(x) = e^x$, which is verified immediately.

When we take the questions for different kernels, we need two specific formulae of integration, which are as follow:

1. Formula for converting a *multiple integral of order n* into a single ordinary integral of order one: It is given below:

$$\int_a^x u(\xi)\left(d\xi\right)^n = \int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!}u(\xi)d\xi \tag{1.19}$$

[$(d\xi)^n$ is also written as $d\xi^n$]

2. Leibnitz's rule of differentiation under the sign of integration: If $F(x, \xi)$ and $\frac{\partial F}{\partial x}$ are continuous functions of both x and ξ and if the first derivatives of $G(x)$ and $H(x)$ are continuous, then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x, \xi)\,d\xi = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}d\xi + F[x, H(x)]\frac{dH}{dx} - F[x, G(x)]\frac{dG}{dx} \tag{1.20}$$

If $G(x)$ and $H(x)$ are constant functions, then Eq. (1.20) reduces to

$$\frac{d}{dx}\int_G^H F(x,\xi)d\xi = \int_G^H \frac{\partial F}{\partial x}d\xi \tag{1.21}$$

EXAMPLE 1.1: Show that the function $u(x) = e^x\left(2x - \frac{2}{3}\right)$ is a solution of Fredholm integral equation $u(x) + 2\int_0^1 e^{x-\xi}u(\xi)\,d\xi = 2xe^x$.

Solution: We have $u(x) = e^x\left(2x - \frac{2}{3}\right)$

$$\because \qquad u(\xi) = e^{\xi}\left(2\xi - \frac{2}{3}\right),$$

Then, the L.H.S. of the given integral equation

$$= e^x\left(2x - \frac{2}{3}\right) + 2\int_0^1 e^{x-\xi}\cdot e^{\xi}\left(2\xi - \frac{2}{3}\right)d\xi$$

$$= e^x\left(2x - \frac{2}{3}\right) + 2e^x\left[\xi^2 - \frac{2\xi}{3}\right]_0^1$$

$$= e^x\left(2x - \frac{2}{3}\right) + 2e^x\left[1 - \frac{2}{3}\right] = 2xe^x$$

$\therefore$ L.H.S. = R.H.S.

Hence proved.

EXAMPLE 1.2: Show that the function $u(x) = xe^x$ is a solution of the Volterra integral equation.

$$u(x) = \sin x + 2\int_0^x \cos(x-\xi) \cdot u(\xi) d\xi$$

Solution: We have $u(x) = xe^x$

$$\therefore \qquad u(\xi) = \xi \cdot e^{\xi}$$

Then, the R.H.S. of given integral equation $= \sin x + 2\int_0^x \cos(x-\xi) \cdot \xi \cdot e^{\xi} d\xi$

$$= \sin x + 2\int_0^x \xi \cdot e^{\xi} \{\cos(\xi - x)\} d\xi$$

(Now, integrating by parts)

$$= \sin x + 2 \cdot \left[\xi \cdot \frac{e^{\xi}}{1+1} \{\cos(\xi - x) + \sin(\xi - x)\} \right]_0^x$$

$$-2 \cdot \int_0^x 1 \cdot \frac{e^{\xi}}{1+1} \{\cos(\xi - x) + \sin(\xi - x)\} d\xi$$

$$= \sin x + xe^x - \int_0^x e^{\xi} \cos(\xi - x) d\xi - \int_0^x e^{\xi} \sin(\xi - x) d\xi$$

$$= \sin x + xe^x - \left[\frac{e^{\xi}}{2} \{\cos(\xi - x) + \sin(\xi - x)\} \right]_0^x$$

$$- \left[\frac{e^{\xi}}{2} \{\sin(\xi - x) - \cos(\xi - x)\} \right]_0^x$$

$$= \sin x + xe^x - \left[\frac{e^x}{2} - \frac{1}{2}(\cos x - \sin x) \right] - \left[\frac{e^x}{2}(-1) - \frac{1}{2}(-\sin x - \cos x) \right]$$

$= xe^x = u(x)$

$\therefore$ L.H.S. = R.H.S.

Hence proved.

EXAMPLE 1.3: Show that $u(x) = \cos 2x$ is a solution of the integral equation

$$u(x) = \cos x + 3\int_0^x K(x, \xi) \cdot u(\xi) d\xi \qquad \text{(i)}$$

where,

$$K(x, \xi) = \begin{cases} \sin x . \cos \xi, & 0 \le x \le \xi \\ \cos x \cdot \sin \xi, & \xi \le x \le \pi \end{cases} \qquad \text{(ii)}$$

Solution: We have $u(x) = \cos 2x$, which means $u(\xi) = \cos 2\xi$, and thus, the R.H.S. of Eq. (i) shapes as [we use kernel as defined by Eq. (ii)]

$$= \cos x + 3\int_{\xi=0}^{\xi=x} \cos x \cdot \sin\xi \cdot \cos 2\xi d\xi + 3\int_{\xi=x}^{\pi} \sin x \cdot \cos\xi \cdot \cos 2\xi \, d\xi$$

$$= \cos x + 3\cos x \int_0^x (\cos 2\xi \cdot \sin\xi)\, d\xi + 3\sin x \int_x^{\pi} (\cos 2\xi \cdot \cos\xi)\, d\xi$$

$$= \cos x + \frac{3}{2}\cos x \int_0^x (\sin 3\xi - \sin\xi)\, d\xi + \frac{3}{2}\sin x \cdot \int_x^{\pi} (\cos 3\xi + \cos\xi)\, d\xi$$

$$= \cos x + \frac{3}{2}\cos x \left[-\frac{1}{3}\cos 3\xi + \cos\xi\right]_0^x + \frac{3}{2}\sin x \left[\frac{1}{3}\sin 3\xi + \sin\xi\right]_x^{\pi}$$

$$= \cos x + \frac{3}{2}\cos x \left[-\frac{1}{3}\cos 3x + \cos x + \frac{1}{3} - 1\right]$$

$$+ \frac{3}{2}\sin x \left[\frac{1}{3}\sin 3\pi + \sin\pi - \frac{1}{3}\sin 3x - \sin x\right]$$

$$= \cos x - \frac{1}{2}(\cos 3x \cdot \cos x + \sin 3x \cdot \sin x) + \frac{3}{2}(\cos^2 x - \sin^2 x) - \cos x$$

$$= -\frac{1}{2}\cos 2x + \frac{3}{2}\cos 2x = \cos 2x = u(x)$$

Hence, $u(x) = \cos 2x$ is a solution of the given integral equation.

EXAMPLE 1.4: Show that the function $u(x) = \sin\left(\frac{\pi x}{2}\right)$ is a solution of the Fredholm integral equation $u(x) - \frac{\pi^2}{4}\int_0^1 K(x, \xi) u(\xi)\, d\xi = \frac{x}{2}$, where the kernel is of the form

$$K(x, \xi) = \begin{cases} x(2-\xi)/2, & 0 \le x \le \xi, \\ \xi(2-x)/2, & \xi \le x \le 1 \end{cases}$$

Solution: Since $u(x) = \sin\left(\frac{\pi x}{2}\right)$, we have $u(\xi) = \sin\left(\frac{\pi\xi}{2}\right)$

L.H.S. of the integral equation

$$= \sin\frac{\pi x}{2} - \frac{\pi^2}{4}\left[\int_0^x K(x,\xi) \cdot u(\xi)\, d\xi + \int_x^1 K(x, \xi) \cdot u(\xi)\, d\xi\right]$$

Now, substituting for kernel $K(x, \xi)$ and $u(\xi)$, we get

$$= \sin\frac{\pi x}{2} - \frac{\pi^2}{4}\left[\int_0^x \left\{\frac{1}{2}\xi(2-x)\right\}\sin\frac{\pi\xi}{2} d\xi + \int_x^1 \left\{\frac{1}{2}x(2-\xi)\right\}\sin\frac{\pi\xi}{2} d\xi\right]$$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2}{8}(2-x)\int_0^x \xi \sin\frac{\pi\xi}{2} d\xi - \frac{\pi^2}{8} x \int_x^1 (2-\xi)\sin\frac{\pi\xi}{2} d\xi$$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2}{8}(2-x)\left[\left[\xi\left\{-\frac{\cos(\pi\xi/2)}{\pi/2}\right\}\right]_0^x - \int_0^x 1 \cdot \left\{-\frac{\cos(\pi\xi/2)}{\pi/2}\right\} d\xi\right]$$

$$-\frac{\pi^2 x}{8}\left[\left[(2-\xi)\left\{-\frac{\cos(\pi\xi/2)}{\pi/2}\right\}\right]_x^1 - \int_x^1 (-1)\left\{\frac{-\cos(\pi\xi/2)}{\pi/2}\right\} d\xi\right]$$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2(2-x)}{8}\left[\frac{-2x}{\pi}\cos\frac{\pi x}{2} + \frac{2}{\pi}\cdot\frac{2}{\pi}\left\{\sin\frac{\pi\xi}{2}\right\}_0^x\right]$$

$$-\frac{\pi^2 x}{8}\left[\frac{2}{\pi}(2-x)\cos\frac{\pi x}{2} - \frac{2}{\pi}\cdot\frac{2}{\pi}\left\{\sin\frac{\pi\xi}{2}\right\}_x^1\right]$$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2(2-x)}{8}\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2} + \frac{4}{\pi^2}\sin\frac{\pi x}{2}\right]$$

$$-\frac{\pi^2 x}{8}\left[\frac{2}{\pi}(2-x)\cos\frac{\pi x}{2} - \frac{4}{\pi^2}\cdot 1 + \frac{4}{\pi^2}\sin\frac{\pi x}{2}\right]$$

$$= \left(\sin\frac{\pi x}{2}\right)\left[1 - \frac{(2-x)}{2} - \frac{x}{2}\right] + \frac{x}{2} = \frac{x}{2}$$

Thus, $u(x) = \frac{x}{2}$ is a solution of the given integral equation.

EXERCISE 1.1

Verify that the given functions are solutions of the corresponding integral equations:

1. $u(x) = (1+x^2)^{-3/2}$, $u(x) = \dfrac{1}{1+x^2} - \displaystyle\int_0^x \frac{\xi}{1+x^2} u(\xi)\, d\xi$

2. $u(x) = 1$, $u(x) + \displaystyle\int_0^1 x(e^{x\xi} - 1)u(\xi)\, d\xi = e^x - x$

3. $u(x) = \dfrac{1}{\pi\sqrt{x}}$, $\displaystyle\int_0^x \frac{u(\xi)}{\sqrt{x-\xi}} d\xi = 1$

4. $u(x) = \sqrt{x}$, $u(x) - \displaystyle\int_0^1 K(x, \xi) u(\xi)\, d\xi$

$$= \sqrt{x} + \frac{x}{15}(4x^{3/2} - 7),\ K(x, \xi) = \begin{cases} \dfrac{x(2-\xi)}{2}, & 0 \le x \le \xi \\ \dfrac{\xi(2-x)}{2}, & \xi \le x \le 1 \end{cases}$$

5. $g(x) = x - \dfrac{x^3}{6}$; $g(x) = x - \displaystyle\int_0^x \sinh(x-t)\cdot g(t)\, dt$

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CHAPTER 2

Applications to Ordinary Differential Equations

2.1 INTRODUCTION

The quest for establishing a representation formula to replace an ordinary differential equation (with initial value problem or boundary value problem) always leads to an integral equation. More specifically, it is found that an initial value problem is converted to a Volterra integral equation, while a boundary value problem is converted to a Fredholm integral equation. Finally, it eases our work of finding the solution of integral equation thus obtained.

Recalling once more, in an initial value problem, the boundary conditions are for the same value of independent variable, while in the case of boundary value problem, the boundary conditions are for different values of independent variable.

2.2 METHOD OF CONVERSION OF AN INITIAL VALUE PROBLEM TO A VOLTERRA INTEGRAL EQUATION

Let an ordinary differential equation of order n be

$$\frac{d^n y}{dx^n} + \alpha_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \alpha_2(x)\cdot\frac{d^{n-2}y}{dx^{n-2}} + \cdots + \alpha_n(x)\cdot y = \phi(x) \tag{2.1}$$

with the initial conditions

$$y(a) = q_0, y'(a) = q_1, \ldots, y^{(n-1)}(a) = q_{n-1} \tag{2.2}$$

where $\alpha_1(x)$, $\alpha_2(x)$, ..., $\alpha_n(x)$, $\phi(x)$ are defined and are continuous in $a \le x \le b$.

Let $u(x)$ be an unknown function such that

$$\frac{d^n y}{dx^n} = u(x) \tag{2.3}$$

Integrating Eq. (2.3) from a to x, we get

$$\left[\frac{d^{n-1}y}{dx^{n-1}}\right]_a^x = \int_a^x u(x)\,dx$$

or $$\frac{d^{n-1}y}{dx^{n-1}} - y^{(n-1)}(a) = \int_a^x u(x)\,dx$$

or $$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^x u(x)\,dx + q_{n-1} \quad \text{[using Eq. (2.2)]} \tag{2.4}$$

or $$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^x u(t)\,dt + q_{n-1} \tag{2.5}$$

Integrating Eq. (2.4) with respect to x from a to x, we obtain

$$\left[\frac{d^{n-2}y}{dx^{n-2}}\right]_a^x = \int_a^x u(x)(dx)^2 + \int_a^x q_{n-1}\,dx$$

or $$\frac{d^{n-2}y}{dx^{n-2}} - y^{(n-2)}(a) = \int_a^x u(x)(dx)^2 + q_{n-1}(x-a)$$

or $$\frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(x)(dx)^2 + (x-a)q_{n-1} + q_{n-2} \quad \text{[using Eq. (2.2)]} \tag{2.6}$$

or $$\frac{d^{n-2}y}{dx^{n-2}} = \int_a^x u(t)(dt)^2 + (x-a)q_{n-1} + q_{n-2}$$

Now, using Eq. (1.19) (i) for the double integral, we get

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_a^x (x-t)\,u(t)\,dt + (x-a)\,q_{n-1} + q_{n-2} \tag{2.7}$$

Now, integrating Eq. (2.6) with respect to x from a to x

$$\left[\frac{d^{n-3}y}{dx^{n-3}}\right]_a^x = \int_a^x u(x)(dx)^3 + q_{n-1}\int_a^x (x-a)dx + q_{n-2}\int_a^x dx$$

or $$\frac{d^{n-3}y}{dx^{n-3}} - y^{(n-3)}(a) = \int_a^x u(x)(dx)^3 + q_{n-1}\left[\frac{(x-a)^2}{2}\right]_a^x + q_{n-2}[x]_a^x$$

or $$\frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(x)(dx)^3 + q_{n-1}\frac{(x-a)^2}{2!} + q_{n-2}(x-a) + q_{n-3} \tag{2.8}$$

or $$\frac{d^{n-3}y}{dx^{n-3}} = \int_a^x u(t)(dt)^3 + q_{n-1}\frac{(x-a)^2}{2!} + (x-a)q_{n-2} + q_{n-3}$$

Now, applying Eq. (1.19) for the triple integral, we get

$$\frac{d^{n-3}y}{dx^{n-3}} = \int_a^x \frac{(x-t)^2}{2!}u(t)dt + q_{n-1}\frac{(x-a)^2}{2!} + q_{n-2}\frac{(x-a)}{1!} + q_{n-3} \tag{2.9}$$

Proceeding in this way, we get it reduced to

$$\frac{dy}{dx}=\int_a^x \frac{(x-t)^{n-2}}{(n-2)!}u(t)dt+q_{n-1}\frac{(x-a)^{n-2}}{(n-2)!}+q_{n-2}\frac{(x-a)^{n-3}}{(n-3)!}+\cdots+q_2(x-a)+q_1 \tag{2.10}$$

and
$$y=\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}u(t)dt+q_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}+q_{n-2}\frac{(x-a)^{n-2}}{(n-2)!}$$
$$+\cdots+q_1(x-a)+q_0 \tag{2.11}$$

Now, multiplying Eqs. (2.3), (2.5), (2.7), (2.9), ..., (2.11) by 1, $\alpha_1(x)$, $\alpha_2(x)$, ..., $\alpha_{n-1}(x)$ and $\alpha_n(x)$, respectively, and adding, we get

$$\frac{d^n y}{dx^n}+\alpha_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\alpha_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\cdots+\alpha_{n-1}(x)\frac{dy}{dx}+\alpha_n(x)\cdot y$$
$$=u(x)+[q_{n-1}\alpha_1(x)+\left\{q_{n-2}+(x-a)q_{n-1}\right\}\alpha_2(x)+\cdots$$
$$+\left\{q_0+q_1(x-a)+\cdots+q_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}\right\}\alpha_n(x)]$$
$$+\int_a^x\left[\alpha_1(x)+(x-t)\alpha_2(x)+\frac{(x-t)^2}{2!}\alpha_3(x)+\cdots+\frac{(x-t)^{n-1}}{(n-1)!}\alpha_n(x)\right]u(t)dt \tag{2.12}$$

Now for L.H.S., we use Eq. (2.1) and let

$$\psi(x)=q_{n-1}\alpha_1(x)+\left\{q_{n-2}+(x-a)q_{n-2}\right\}\alpha_2(x)+\cdots$$
$$+\left\{q_0+q_1(x-a)+\cdots+q_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}\right\}\alpha_n(x)$$

and
$$K(x,t)=-\left[\alpha_1(x)+(x-t)\alpha_2(x)+\cdots+\frac{(x-t)^{n-1}}{(n-1)!}\alpha_n(x)\right]$$

so that Eq. (2.12) reduces to

$$\phi(x)=u(x)+\psi(x)-\int_a^x K(x,t)\cdot u(t)dt \tag{2.13}$$

Again, let $\phi(x)-\psi(x)=f(x)$, then

$$u(x)=f(x)+\int_a^x K(x,t)\cdot u(t)dt \tag{2.14}$$

which is Volterra integral equation of the second kind. Thus, we have found a relation between a linear differential equation (Eq. 2.1) and a Volterra integral equation, which establishes that an initial value problem is converted to a Volterra integral equation.

EXAMPLE 2.1: Transform the initial value problem

$$\frac{d^2y}{dx^2}+xy=1, y(0)=0, y'(0)=0$$

to a Volterra integral equation of the second kind.

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + x \cdot y = 1 \tag{i}$$

subject to initial conditions

$$y(0) = 0, \qquad \text{[ii(a)]}$$

$$y'(0) = 0 \qquad \text{[ii(b)]}$$

Let
$$\frac{d^2y}{dx^2} = u(x) \tag{iii}$$

Now, integrating Eq. (iii) with respect to x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)dx \quad \text{or} \quad \frac{dy}{dx} - y'(0) = \int_0^x u(x)dx$$

which on using Eq. [ii(b)], becomes

$$\frac{dy}{dx} = \int_0^x u(x)dx \tag{iv}$$

or
$$\frac{dy}{dx} = \int_0^x u(t)dt \tag{v}$$

Now, integrating Eq. (iv) with respect to x and using Eq. [ii(a)], we obtain

$$y(x) - y(0) = \int_0^x u(x)(dx)^2 \Rightarrow y(x) = \int_0^x u(t)(dt)^2$$

which on applying Eq. (1.19) gives

$$y(x) = \int_0^x (x-t)u(t)dt \tag{vi}$$

Finally, putting $\frac{d^2y}{dx^2}$ from Eq. (iii) and y from Eq. (vi) into Eq. (i), we get

$$u(x) + x\left[\int_0^x (x-t)u(t)dt\right] = 1 \quad \text{or} \quad u(x) = 1 - \int_0^x (x-t)u(t)dt$$

which is the required equation.

EXAMPLE 2.2: Transform the initial value problem

$$\frac{d^3y}{dx^3} - 2xy = 0$$

subject to initial conditions

$$y(0) = \frac{1}{2},\ y'(0) = 1,\ y''(0) = 1$$

into a non-homogeneous Volterra integral equation of the second kind.

Solution:

The given equation is

$$\frac{d^3y}{dx^3} - 2xy = 0 \tag{i}$$

Let
$$\frac{d^3y}{dx^3} = u(x) \tag{ii}$$

Integrating Eq. (ii) with respect to x from 0 to x, we get

$$\left[\frac{d^2y}{dx^2}\right]_0^x = \int_0^x u(x)\,dx \quad \text{or} \quad y''(x) - y''(0) = \int_0^x u(x)\,dx$$

or
$$y''(x) = 1 + \int_0^x u(x)\,dx \tag{iii}$$

Again, integrating Eq. (iii) with respect to x from 0 to x, we obtain

$$[y'(x)]_0^x = \int_0^x dx + \int_0^x u(x)(dx)^2$$

or
$$y'(x) = 1 + x + \int_0^x u(x)(dx)^2 \tag{iv}$$

Integrating once more with respect to x from 0 to x, we get

$$y(x) - y(0) = \int_0^x dx + \int_0^x x\ dx + \int_0^x u(x)(dx)^3$$

or
$$y(x) = \frac{1}{2} + x + \frac{x^2}{2} + \int_0^x u(t)(dt)^3$$

or
$$y(x) = \left(\frac{1}{2} + x + \frac{x^2}{2}\right) + \int_0^x \frac{(x-t)^2}{2!} u(t)dt \quad \text{[using Eq. (1.19)} \tag{v}$$

Now, putting the values from Eqs. (ii) and (v) in Eq. (i), we get

$$u(x) - 2x\left(\frac{1}{2} + x + \frac{x^2}{2}\right) - 2x\int_0^x \frac{(x-t)^2}{2!} u(t)dt = 0$$

or $u(x) = x(x+1)^2 + \int_0^x x(x-t)^2 u(t)dt$, which is the required equation.

Note: In this example, we have not changed the variable of integration from x to t for $y''(x)$ and $y'(x)$, since this change was not required.

EXAMPLE 2.3: Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + e^x \cdot y = x$$

with the initial conditions

$$y(0) = 1,\ y'(0) = -1$$

Solution:

The given equation is

$$\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + e^x \cdot y = x \tag{i}$$

Let

$$\frac{d^2y}{dx^2} = u(x) \tag{ii}$$

Integrating Eq. (ii) with respect to x from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)dx \;\text{ or }\; \frac{dy}{dx} - y'(0) = \int_0^x u(x)dx$$

or
$$\frac{dy}{dx} = -1 + \int_0^x u(x)dx \tag{iii}$$

or
$$\frac{dy}{dx} = -1 + \int_0^x u(t)dt \tag{iv}$$

Integrating Eq. (iii) again with respect to x from 0 to x, we get

$$[y(x)]_0^x = -[x]_0^x + \int_0^x u(x)(dx)^2$$

or
$$y(x) - y(0) = -x + \int_0^x u(x)(dx)^2$$

or
$$y(x) = 1 - x + \int_0^x u(t)(dt)^2$$

or
$$y(x) = 1 - x + \int_0^x (x-t)u(t)dt \tag{v}$$

Putting the values of $y, \dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from Eqs. (v), (iv) and (ii), respectively, in Eq. (i), we get $u(x) = x - \sin x - e^x(1-x) + \int_0^x [\sin x - e^x(x-t)]u(t)\,dt$, which is non-homogeneous Volterra integral equation of second kind.

EXERCISE 2.1

1. Reduce the following initial value problems to Volterra integral equations of the second kind.

   (a) $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 3y = 0, y(0) = 1, y'(0) = 0$

   (b) $y'' + xy' + y = 0$, $y(0) = 1, y'(0) = 0$

(c) $y''' + xy'' + (x^2 - x)y = xe^x + 1$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$

(d) $\dfrac{d^2y}{dx^2} + \alpha_1(x)\dfrac{dy}{dx} + \alpha_2(x)y = F(x), y(0) = c_0, y'(0) = c_1$

Answers:

(a) $u(x) = 3 + \int_0^x (5x - 3t) \cdot u(t)dt$

(b) $u(x) = -1 - \int_0^x (2x - t) \cdot u(t)dt$

(c) $u(x) = xe^x + 1 - x(x^2 - 1) - \int_0^x \left\{ x + \frac{1}{2}(x^2 - x)(x - t)^2 \right\} u(t)dt$

(d) $u(x) = F(x) - c_1\alpha_1(x) + (c_0 + c_1x)\alpha_2(x) - \int_0^x \{\alpha_1(x) + \alpha_2(x).(x - t)\} u(t)dt$

2.3 ALTERNATE METHOD OF TRANSFORMING THE INITIAL VALUE PROBLEM INTO A VOLTERRA INTEGRAL EQUATION

We can convert a linear differential equation describing an initial value problem into a Volterra integral equation by a different procedure also. Another advantage with this method is that we can derive the original differential equation with its initial conditions from the transformed integral equation. The procedure will be clear through the examples given here.

EXAMPLE 2.4: Transform the following initial value problem into an integral equation:

$$\frac{d^2y}{dx^2} + \alpha(x)\frac{dy}{dx} + \beta(x)y = \gamma(x), \quad y(a) = a_0, y'(a) = b_0$$

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} = \gamma(x) - \beta(x) \cdot y(x) - \alpha(x)\frac{dy}{dx} \tag{i}$$

and the given initial conditions are

$$y(a) = a_0 \tag{ii(a)}$$

$$y'(a) = b_0 \tag{ii(b)}$$

Integrating with respect to x from a to x

$$\left[\frac{dy}{dx}\right]_a^x = \int_a^x \{\gamma(x) - \beta(x)y(x)\} dx - \int_a^x \left\{\alpha(x)\frac{dy}{dx}\right\} dx$$

Now, integrating by parts,

or $\quad \dfrac{dy}{dx} - b_0 = \int_a^x \{\gamma(x) - \beta(x)y(x)\} dx - [\alpha(x) \cdot y(x)]_a^x + \int_a^x \alpha'(x) \cdot y(x)\, dx$

Now, using Eqs. [ii(a)] and [ii(b)],

or $$\frac{dy}{dx} = b_0 - \alpha(x)y(x) + \alpha(a)\cdot a_0 + \int_a^x \{\gamma(x) - \beta(x)y(x) + \alpha'(x)y(x)\}\,dx$$

Integrating again with respect to x from a to x,

$$[y]_a^x = \int_a^x \{b_0 + \alpha(a)\cdot a_0\}\,dx - \int_a^x \alpha(x)\cdot y(x)\,dx$$
$$+ \int_a^x \{\gamma(x) - \beta(x)y(x) + \alpha'(x)\cdot y(x)\}(dx)^2$$

or $$y(x) - y(a) = \{b_0 + \alpha(a)\cdot a_0\}(x-a) - \int_a^x \alpha(t)\cdot y(t)\,dt$$
$$+ \int_a^x \{\gamma(t) - \beta(t)\cdot y(t) + \alpha'(t)\cdot y(t)\}(dt)^2$$

Now, using Eq. (1.19), we get

$$y(x) = a_0 + (x-a)\{b_0 + \alpha(a)\cdot a_0\} - \int_a^x \alpha(t)\cdot y(t)\,dt$$
$$+ \int_a^x (x-t)\{\gamma(t) - \beta(t)y(t) + \alpha'(t)y(t)\}(dt)$$

or $$y(x) = a_0 + (x-a)\{b_0 + a_0\cdot\alpha(a)\} + \int_0^x (x-t)\gamma(t)\,dt$$
$$- \int_a^x [\alpha(t) + (x-t)\{\beta(t) - \alpha'(t)\}]\,y(t)\,dt$$

This is the required integral equation.

EXAMPLE 2.5: Convert $y'' - \sin x\cdot y' + e^x\cdot y = x$ with initial conditions $y(0)=1,\ y'(0)=-1$ to a Volterra equation of the second kind. Also, derive the original differential equation with the initial conditions from the integral equation obtained in the previous part.

Solution: The given differential equation with the initial conditions is

$$y''(x) = x - e^x\cdot y(x) + \sin x\cdot y'(x) \tag{i}$$

with $$y(0) = 1, \tag{ii(a)}$$
$$y'(0) = -1 \tag{ii(b)}$$

Integrating Eq. (i) with respect to x from 0 to x, we get

$$[y'(x)]_0^x = \frac{x^2}{2} - \int_0^x e^x y(x)\,dx + \int_0^x \sin x\cdot y'(x)\,dx$$

or $$y'(x) + 1 = \frac{x^2}{2} - \int_0^x e^x\cdot y(x)\,dx + [\sin x\cdot y(x)]_0^x - \int_0^x \cos x\cdot y(x)\,dx$$

or $$y'(x) = \frac{x^2}{2} - 1 + \sin x\cdot y(x) - \int_0^x (e^x + \cos x)\cdot y(x)\,dx \tag{iii}$$

Integrating Eq. (iii) with respect to x from 0 to x, we get

$$[y(x)]_0^x = \left[\frac{x^3}{6} - x\right]_0^x + \int_0^x \sin t\, y(t)\, dt - \int_0^x (e^t + \cos t) y(t)(dt)^2$$

or
$$y(x) - 1 = \frac{x^3}{6} - x + \int_0^x [\sin t - (x-t)(e^t + \cos t)] y(t)\, dt$$

or
$$y(x) = \left(\frac{x^3}{6} - x + 1\right) + \int_0^x [\sin t - (x-t)(e^t + \cos t)] y(t)\, dt \qquad \text{(iv)}$$

which is a Volterra integral equation of the second kind.

Now, we differentiate Eq. (iv) with respect to x and obtain

$$y'(x) = \frac{x^2}{2} - 1 + \frac{d}{dx}\int_0^x \{\sin t - (x-t)(e^t + \cos t)\} y(t)\, dt$$

For this, we apply Leibnitz's rule of differentiation under the sign of integration Eq. (1.20), and get

$$y'(x) = \frac{x^2}{2} - 1 + \int_0^x \frac{\partial}{\partial x}[\{\sin t - (x-t)(e^t + \cos t)\} y(t)]\, dt$$
$$+ [\sin x - (x-x)(e^x + \cos x)] y(x) \frac{d}{dx}(x)$$
$$-[\sin 0 - (x-0)(e^0 + \cos 0)] y(0) \cdot \frac{d}{dx}(0)$$

or
$$y'(x) = \frac{x^2}{2} - 1 - \int_0^x [e^t + \cos t] y(t)\, dt + \sin x \cdot y(x) \qquad \text{(v)}$$

Differentiating Eq. (v) with respect to x

$$y''(x) = x + \cos x \cdot y(x) + \sin x \cdot y'(x) - \frac{d}{dx}\int_0^x (e^t + \cos t) y(t)\, dt$$

or
$$y''(x) = x + \cos x \cdot y(x) + \sin x \cdot y'(x) - \left[\int_0^x \frac{\partial}{\partial x}\{e^t + \cos t) y(t)\}\, dt\right.$$
$$\left. + (e^x + \cos x) \cdot y(x) \cdot \frac{d(x)}{dx} - (e^0 + \cos 0) y(0) \cdot \frac{d(0)}{dx}\right]$$

or
$$y''(x) = x + \cos x \cdot y(x) + \sin x \cdot y'(x) - [0 + (e^x + \cos x) y(x) \cdot 1 + 0]$$

or
$$y''(x) - \sin x \cdot y'(x) + e^x y(x) = x \qquad \text{(vi)}$$

which is same as the given differential equation. To get the initial conditions, we put $x = 0$ in Eq. (iv) and (v), and obtain

$$y(0) + 1 \text{ and } y'(0) = -1 \qquad \text{(vii)}$$

Thus Eqs. (vi) and (vii) together provide the original initial value problem.

EXERCISE 2.2

1. Convert the following initial value problems into integral equations.

   (a) $\dfrac{d^2y}{dx^2} + nxy = g(x),\ y(0) = 1, y'(0) = 0$

   (b) $\dfrac{d^2y}{dx^2} + y = 0, \qquad y(0) = 0, y'(0) = 0$

2. Convert $y''(x) - 3y'(x) + 2y(x) = 4\sin x$ with initial conditions $y(0) = 1,\ y'(0) = -2$ into a Volterra integral equation. Also, find the original initial value problem from the obtained integral equation.
3. Transform $y'' + xy = 1,\ y(0) = 0,\ y(1) = 1$ into an integral equation.
4. Transform the boundary value problem $y'' + y = x,\ y(0) = 0,\ y'(1) = 0$ to a Fredholm integral equation.

Answers:

1. (a) $y(x) = 1 + \int_0^x (x-t)[g(t) - nt\,y(t)]\,dt$

   (b) $y(x) = -\int_0^x (x-t)\,y(t)\,dt$

2. $y(x) = 1 - x - 4\sin x + \int_o^x \{3 - 2(x-t)\}y(t)\ dt$
3. $y(x) = \dfrac{1}{2}x(1+x) + \int_0^1 K(x, t)\,y(t)\,dt; K(x, t) = \begin{cases} t^2(1-x),\ t < x \\ xt(1-t),\ t > x \end{cases}$
4. $y(x) = \dfrac{x^3}{6} - \dfrac{x}{2} + \int_0^1 K(x, t)\ y(t)\,dt;\ K(x, t) = \begin{cases} t,\ t < x \\ x,\ t > x \end{cases}$

2.4 BOUNDARY VALUE PROBLEM AND ITS CONVERSION TO FREDHOLM INTEGRAL EQUATION

When an ordinary differential equation is given with the conditions involving dependent variable and its derivatives at two different values of independent variables, the problem under consideration is said to be a *boundary value problem*.

The method of conversion of a boundary value problem to a Fredholm integral equation can be made clear by the examples given hereunder.

EXAMPLE 2.6: Reduce the following boundary value problem into an integral equation:

$$\frac{d^2u}{dx^2} + \lambda u = 0 \text{ with } u(0) = 0, u(l) = 0$$

Solution:

The given conditions are

$$u(0) = 0 \qquad \text{[i(a)]}$$

$$u(l) = 0 \qquad \text{[i(b)]}$$

The given differential equation is

$$u(x) = -\lambda u(x)$$

Integrating it with respect to x from 0 to x, we get

$$\int_0^x u''(x)\,dx = -\lambda \int_0^x u(x)\,dx$$

or
$$u'(x) - u'(0) = -\lambda \int_0^x u(x)\,dx$$

Let $\quad u'(0) = c \quad$ so that $\; u'(x) = c - \lambda \int_0^x u(x)\,dx.$

Again, on integration, we get

$$\int_0^x u'(x)\,dx = c\int_0^x dx - \lambda \int_0^x u(x)(dx)^2 = cx - \lambda \int_0^x u(t)(dt)^2$$

or
$$u(x) - u(0) = cx - \lambda \int_0^x (x-t)\cdot u(t)\,dt$$

Now using Eq. [i(a)],

$$u(x) = cx - \lambda \int_0^x (x-t)u(t)\,dt \qquad \text{(ii)}$$

Now, to determine c, we use Eq. [i(b)] by taking $x = 1$, so that

$$u(l) = 0 = cl - \lambda \int_0^l (l-t)u(t)\,dt$$

$\Rightarrow$
$$c = \frac{\lambda}{l}\int_0^l (l-t)u(t)\,dt$$

Now, Eq. (ii) can be expressed as

$$u(x) = \frac{\lambda}{l}x\int_0^l (l-t)u(t)\,dt - \lambda \int_0^x (x-t)u(t)\,dt$$

$$u(x) = \int_0^l \frac{\lambda x(l-t)}{l}u(t)\,dt - \int_0^x \lambda(x-t)u(t)\,dt$$

or
$$u(x) = \int_0^x \frac{\lambda x(l-t)}{l}u(t)\,dt + \int_x^l \frac{\lambda x(l-t)}{l}u(t)\,dt - \int_0^x \lambda(x-t)u(t)\,dt$$

or
$$u(x) = \lambda \int_0^x \left\{\frac{x(l-t)}{l} - (x-t)\right\}u(t)\,dt + \lambda \int_x^l \frac{x(l-t)}{l}u(t)\,dt$$

or
$$u(x) = \lambda\left[\int_0^x \frac{t(l-x)}{l}\,u(t)\,dt + \int_x^l \frac{x(l-t)}{l}\,u(t)\,dt\right]$$

or $$u(x) = \lambda \int_0^l K(x, t) \cdot u(t)\, dt \tag{iii}$$

or where, $$K(x, t) = \begin{cases} \dfrac{t(l-x)}{l}, & \text{if } 0 < t < x \\ \dfrac{x(l-t)}{l}, & \text{if } x < t < l \end{cases} \tag{iv}$$

Equation (iii) is the required integral equation, whose kernel $K(x, t)$ is defined by Eq. (iv).

EXAMPLE 2.7: Transform the boundary value problem

$$\frac{d^2y}{dx^2} + y = x, y(0) = 0, y'(1) = 0$$

to a Fredholm integral equation. Also, recover the boundary value problem from the integral equation.

Solution:

The given conditions are

$$y\ (0) = 0 \tag{[i(a)]}$$

$$y'(1) = 0 \tag{[i(b)]}$$

Integrating the given differential equation with respect to x, we have

$$\int_0^x y''(x)\, dx + \int_0^x y(x)\, dx = \int_0^x x\, dx$$

or $$y'(x) - y'0) = \frac{x^2}{2} - \int_0^x y(x)\, dx$$

Let $y'(0) = c$, so that

$$y'(x) = \left(c + \frac{x^2}{2}\right) - \int_0^x y(x)\, dx \tag{ii}$$

Integrating both sides with respect to x

$$y(x) - y(0) = cx + \frac{x^3}{6} - \int_0^x y(x)(dx)^2 \tag{iii}$$

Using Eq. [i(a)] we express Eq. (iii) as

$$y(x) = cx + \frac{x^3}{6} - \int_0^x y(t)(dt)^2$$

$\Rightarrow$ $$y(x) = cx + \frac{x^3}{6} - \int_0^x (x-t)\, y(t)\, dt \tag{[iv(a)]}$$

Now, to determine c, we differentiate [iv(a)] and use Eq. [i(b)].

$$0 = c + \frac{1}{2} - \int_0^1 y(t)\,dt$$

[Here, we apply Eq. (1.20)]

$\Rightarrow$ $$c = -\frac{1}{2} + \int_0^1 y(t)\,dt$$

and then, Eq. [iv(a)] is

$$y(x) = x\left[-\frac{1}{2} + \int_0^1 y(t)\,dt\right] + \frac{x^3}{6} - \int_0^x (x-t)\,y(t)\,dt \qquad \text{[iv(b)]}$$

$$y(x) = -\frac{x}{2} + \frac{x^3}{6} + \left\{\int_0^x x.y(t)\,dt + \int_x^1 x.y(t)\,dt\right\} - \int_0^x (x-t)\,y(t)\,dt$$

$$y(x) = \frac{x^3 - 3x}{6} + \int_0^x t \cdot y(t)dt + \int_x^1 x \cdot y(t)\,dt$$

or $$y(x) = \frac{1}{6}(x^3 - 3x) + \int_0^1 K(x, t)\,y(t)\,dt \qquad \text{(v)}$$

where, $$K(x, t) = \begin{cases} x, & x < t \\ t, & x > t \end{cases} \qquad \text{(vi)}$$

Converse: We take the integral equation, i.e., Eq. [iv(b)].

$$y(x) = -\frac{x}{2} + \frac{x^3}{6} + \int_0^1 x \cdot y(t)\,dt - \int_0^x (x-t)\,y(t)\,dt \qquad \text{(vii)}$$

Differentiating both sides with respect to x, we have

$$y'(x) = -\frac{1}{2} + \frac{x^2}{2} + \frac{d}{dx}\int_0^1 x \cdot y(t)\,dt - \frac{d}{dx}\int_0^x (x-t)\,y(t)\,dt$$

$$y'(x) = -\frac{1}{2} + \frac{x^2}{2} + \int_0^1 y(t)\,dt - \int_0^x y(t)\,dt \qquad \text{(viii)}$$

Again, differentiating with respect to x, we get

$$y''(x) = x + \frac{d}{dx}\int_0^1 y(t)\,(dt) - \frac{d}{dx}\int_0^x y(t)\,(dt)$$

$$y''(x) = x + 0 - \int_0^x \frac{\partial}{\partial x}\{y(t)\}\,dt - y(x)$$

$$y''(x) = x - 0 - y(x)$$

or $$y''(x) + y = x \qquad \text{(ix)}$$

Also from Eqs. (vii) and (viii)

$$y(0) = 0 \text{ and } y'(1) = 0$$

So, Eq. (ix) is the original differential equation with the boundary conditions.

EXAMPLE 2.8: Transform $y'' + xy = 1$, $y(0) = 0$, $y(1) = 1$ into an integral equaton.

Solution: The given differential equation is

$$y'' = 1 - xy \tag{i}$$

with the boundary conditions

$$y(0) = 0 \tag{ii(a)}$$

$$y(1) = 1 \tag{ii(b)}$$

Integrating Eq. (i) with respect to x (x varying from 0 to x),

$$\int_0^x y''(x)\,dx = \int_0^x dx - \int_0^x x \cdot y(x)\,dx$$

$$\Rightarrow \qquad y'(x) - y'(0) = x - \int_0^x x \cdot y(x)\,dx$$

Let $\qquad y'(0) = c$, so that

$$y'(x) - c = x - \int_0^x x \cdot y(x)\,dx$$

Integrating once more with respect to x from 0 to x,

$$\int_0^x y'(x)\,dx = \int_0^x (c + x)\,dx - \int_0^x x \cdot y(x)(dx)^2$$

or $$y(x) - y(0) = \left[cx + \frac{x^2}{2}\right]_0^x - \int_0^x t \cdot y(t)dt)^2$$

or $$y(x) - 0 = \frac{1}{2}(2cx + x^2) - \int_0^x (x - t)\,t\,y(t)\,dt \tag{iii}$$

Now, to determine c, we use Eq. [ii(b)], which is $y(1) = 1$.

$$1 = \frac{1}{2}(2c + 1) - \int_0^1 (1 - t) \cdot t \cdot y(t)\,dt$$

$$\therefore \qquad c = \frac{1}{2} + \int_0^1 (1 - t) \cdot t \cdot y(t)\,dt \tag{iv}$$

Putting the value of c in Eq. (iii), we get

$$y(x) = x\left[\frac{1}{2} + \int_0^1 (1 - t)t \cdot y(t)\,dt\right] + \frac{x^2}{2} - \int_0^x t(x - t) \cdot y(t)\,dt$$

$$y(x) = \frac{x}{2}(1 + x) + \int_0^x xt(1 - t) \cdot y(t)\,dt + \int_x^1 xt(1 - t) \cdot y(t)\,dt - \int_0^x t(x - t) \cdot y(t)\,dt$$

$$y(x) = \frac{x}{2}(1 + x) + \int_0^x t^2(1 - x) \cdot y(t)\,dt + \int_x^1 xt(1 - t) \cdot y(t)\,dt$$

or $$y(x) = \frac{x}{2}(1+x) + \int_0^x K(x,t)\cdot y(t)\,dt \tag{v}$$

where, $$K(x,\,t) = \begin{cases} t^2(1-x), \text{ when } t < x \\ xt(1-t), \text{ when } t > x \end{cases} \tag{vi}$$

Equation (v) is the required integral equation, whose kernel $K(x,\,t)$ is defined by Eq. (vi).

EXAMPLE 2.9: If $\mu(x)$ is continuous and satisfies

$$u(x) = \int_0^1 \lambda\cdot K(x,\,\xi)\,u(\xi)\,d\xi, \text{ where } K(x,\,\xi) = \begin{cases} (1-\xi)x, \; 0 \le x \le \xi \\ (1-x)\xi, \; \xi \le x \le 1 \end{cases},$$

then prove that $\mu(x)$ is also a solution of the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda u = 0, u(0) = 0,\; u(1) = 0$$

Solution: Here, from the integral equation,

$$u(x) = \lambda \int_0^1 K(x,\,\xi)\cdot u(\xi)\;d\xi \tag{i}$$

where, $$K(x,\,\xi) = \begin{cases} (1-\xi)\,x, \;\; 0 \le x \le \xi \\ (1-x)\xi, \;\; \xi \le x \le 1 \end{cases} \tag{ii}$$

We have to find the differential equation satisfying $u(x)$ with the corresponding boundary conditions. The kernel may be followed by Figure. 2.1 given below:

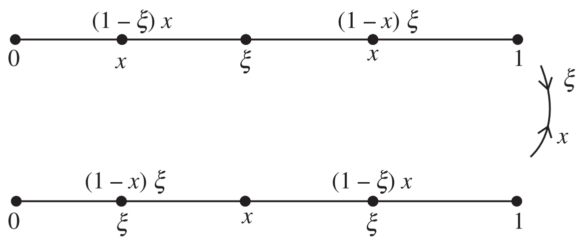

Figure 2.1 Example 2.9.

Equation (i) is now expressed as

$$u(x) = \lambda\left[\int_0^x K(x,\,\xi)\;u(\xi)\,d\xi + \int_x^1 K(x,\,\xi)\;u(\xi)\,d\xi\right]$$

$$u(x) = \lambda\left[\int_0^x (1-x)\xi\cdot u(\xi)\,d\xi + \int_x^1 (1-\xi)x\cdot u(\xi)\,d\xi\right] \tag{iii}$$

Let $x = 0,\ 1$. We find $u(0) = 0$ [iv(a)]
and $u(1) = 0$ [iv(b)]

Now, differentiating Eq. (iii) with respect to x, we get [using Leibnitz's rule Eq. (1.20)]

$$\frac{du}{dx} = \frac{d}{dx}\int_0^x \lambda(1-x)\xi \cdot u(\xi)\,d\xi + \frac{d}{dx}\int_x^1 \lambda x(1-\xi)\cdot u(\xi)\,d\xi$$

$$\frac{du}{dx} = \int_0^x \frac{\partial}{\partial x}\{\lambda(1-x)\xi\cdot u(\xi)\}d\xi + \lambda(1-x)\cdot x\cdot u(x)\cdot\frac{dx}{dx} - 0\cdot\frac{d}{dx}(0)$$

$$+\int_x^1 \frac{\partial}{\partial x}\{\lambda x(1-\xi)\cdot u(\xi)\}\,d\xi + 0\cdot\frac{d}{dx}(1) - \lambda x(1-x)\cdot u(x)\cdot\frac{dx}{dx}$$

or
$$\frac{du}{dx} = \int_0^x \lambda(-1)\cdot\xi\cdot u(\xi)\,d\xi + \int_x^1 \lambda(1-\xi)\cdot u(\xi)\,d\xi \qquad \text{(v)}$$

Differentiating Eq. (v) with respect to x using Leibnitz's rule, we get

$$\frac{d^2u}{dx^2} = -\left[\int_0^x \frac{\partial}{\partial x}\{\lambda\xi\, u(\xi)\}\,d\xi + \lambda x\cdot u(x)\frac{dx}{dx} - 0\right]$$

$$+\int_x^1 \frac{\partial}{\partial x}\{\lambda(1-\xi)u(\xi)\}d\xi + 0 - \lambda(1-x)\cdot u(x)\frac{dx}{dx}$$

or
$$\frac{d^2u}{dx^2} = -\lambda x\, u(x) - \lambda(1-x)\, u(x) \Rightarrow \frac{d^2u}{dx^2} + \lambda u = 0 \qquad \text{(vi)}$$

which is the required equation with boundary conditions given by Eq. [iv(a)] and [iv(b)].

EXERCISE 2.3

1. (a) If $y''(x) + \lambda y(x) = 0$, and y satisfies the end conditions $y(0) = 0$, $y(l) = 0$, show that $y(x) = \frac{\lambda x}{l}\int_0^l (l-t)y(t)\,dt - \lambda\int_0^x (x-t)y(t)\,dt.$

   (b) Show that the result of part (a) may be expressed as

   $$y(x) = \lambda\int_0^l K(x,t)y(t)\,dt$$

   where, $K(x,t) = \begin{cases} \dfrac{t(l-x)}{l}, \text{when } t < l \\ \dfrac{x(l-t)}{l}, \text{when } t > x \end{cases}$

   (c) Verify directly that the expression obtained in part (b) satisfies the prescribed differential equations and end conditions.

2. Convert the problem $y'' + \lambda y = 0;\ y(0) = y'0),\ y(\pi) = y'(\pi)$ to an integral equation.

Answer:

$$y(x) = \lambda \int_0^{\pi} K(x,t) \cdot y(t)dt$$

where,
$$K(x, t) = \begin{cases} \dfrac{(t+1)(\pi - x - 1)}{\pi}, \ 0 \le t \le x \\ \dfrac{(x+1)(\pi - t - 1)}{\pi}, \ x \le t \le \pi \end{cases}$$

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CHAPTER 3

Solution of Homogeneous Fredholm Integral Equations of the Second Kind

3.1 INTRODUCTION

In the previous chapters, we have learnt the basic terminology and suitability of integral equations over differential equations, and then, the classification of the integral equations. Although, there have been many developments in the theory, the basic division as initial value problems into volterra integral equations and boundary value problems into Fredholm integral equations is a must to follow. As we approach to simplify these equations, it is found convenient to focus the kernel, kind and the homogeneous nature of these equations. In this chapter, we shall restrain ourselves only for the homogeneous Fredholm integral equations of the second kind. The chapter begins with the discussion of an essential part called eigenvalue, eigenfunction, and then related theorems are explained.

3.2 CHARACTERISTIC VALUE* (OR EIGENVALUE) AND CHARACTERISTIC FUNCTION (OR EIGENFUNCTION)

We consider the following homogeneous Fredholm integral equation of the second kind:

$$u(x) = \lambda \int_a^b K(x, t) u(t)\, dt \tag{3.1}$$

It is clear that $u(x) = 0$ will always satisfy Eq. (3.1), and we call such $u(x) = 0$ as the *trivial solution* of Eq. (3.1). The values of parameter λ for which the integral equation [Eq. (3.1)] has non-trivial (non-zero) solutions [$u(x) \neq 0$] are known as *eigenvalues* of Eq. (3.1) or characteristic values of kernel $K(x, t)$. If for $u(x) \neq 0$, there exists a continuous function $\phi(x)$ in the interval $[a, b]$ such that

$$\phi(x) = \lambda_0 \int_a^b K(x, t) \phi(t)\, dt \tag{3.2}$$

* It is also called characteristic number.

then $\phi(x)$ is known as *eigenfunction*[*] of Eq. (3.1) corresponding to eigenvalue λ_0.

Note:

1. If kernel $K(x, t)$ is continuous for $a \le x \le b$, $a \le t \le b$, for the finite values of a and b, then corresponding to every eigenvalue of λ there exists a finite number of linearly independent eigenfunctions; the number of such functions is called *index of the eigenvalue*. Different eigenvalues have different indices.
2. As seen above, along with $\phi(x)$, $c\phi(x)$ is also an eigenfunction for λ_0, where c is an arbitrary constant (though not L.I.).
3. The number $\lambda = 0$ is not taken as eigenvalue, since it provides trivial solution.
4. A homogeneous Fredholm integral equation may not have eigenvalues and eigenfunctions if the kernel is not symmetric.

EXAMPLE 3.1: Find the eigenvalue and eigenfunction of the homogeneous integral equation $g(x) = \lambda \int_0^1 e^{x+t} g(t)\, dt$.

Solution: The given integral equation is

$$g(x) = \lambda e^x \int_0^1 e^t g(t)\, dt \tag{i}$$

Let

$$c = \int_0^1 e^t g(t) dt \tag{ii}$$

so that Eq. (i) provides $g(x) = \lambda c e^x$ or $g(t) = \lambda c e^t$ (iii)

Then, Eq. (ii) provides $c = \int_0^1 e^t \cdot \lambda c e^t dt$

$$c = \frac{\lambda c}{2}\left[e^{2t}\right]_0^1 = \frac{\lambda c}{2}(e^2 - 1)$$

which is

$$c\left\{1 - \frac{\lambda}{2}(e^2 - 1)\right\} = 0$$

Now, for non-trivial solutions,

$$1 - \frac{\lambda}{2}(e^2 - 1) = 0$$

$\Rightarrow$

$$\lambda = \frac{2}{e^2 - 1} \tag{iv}$$

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*It is obvious that an eigenfunction will satisfy the integral equation.

This is the eigenvalue of Eq. (i), and using Eq. (iii), the corresponding eigenfunction is

$$g(x) = \frac{2}{e^2 - 1} ce^x = e^x$$

by taking the constant $\dfrac{2c}{e^2 - 1}$ as unity.

EXAMPLE 3.2: Show that the homogeneous integral equation

$$\phi(x) = \lambda \int_0^1 (3x - 2)\, t\phi(t) dt$$

has no characteristic number and no eigenfunction.

Solution: The given integral equation is

$$\phi(x) = \lambda(3x - 2) \int_0^1 t\phi(t) dt \qquad \text{(i)}$$

Let
$$c = \int_0^1 t\phi(t) dt \qquad \text{(ii)}$$

so that
$$\phi(x) = \lambda(3x - 2)c \qquad \text{(iii)}$$

or
$$\phi(t) = \lambda c(3t - 2)$$

Then,
$$c = \int_0^1 t \cdot \lambda c(3t - 2) dt$$

$\Rightarrow$
$$c = \lambda c \Big[t^3 - t^2 \Big]_0^1 \Rightarrow c = 0$$

Thus, $\phi(x) = 0$ is the solution of Eq. (i) and we do not get any characteristic number or eigenfunction.

We also see that kernel $K(x, t) = (3x - 2)t$ is not symmetric, and in this case, the kernel does not possess any characteristic number necessarily.

3.3 SOLUTION OF HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE (OR DEGENERATE) KERNEL

Let the homogeneous Fredholm integral equation of the second kind be

$$u(x) = \lambda \int_a^b K(x,t) u(t) dt \qquad (3.3)$$

where kernel $K(x, t)$ is separable, which means it can be expressed as the sum of the product of terms, each having functions of x and t separately. Let

$$K(x,t) = \sum_{i=1}^{n} f_i(x) g_i(t) \qquad (3.4)$$

Then, Eq. (3.3) is

$$u(x) = \lambda \int_a^b \left[\sum_{i=1}^{n} f_i(x) g_i(t) \right] u(t) dt \qquad (3.5)$$

$$u(x) = \lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t) u(t) dt \tag{3.6}$$

[By interchanging the order of integration and summation]

Let
$$C_i = \int_a^b g_i(t) u(t) dt, \quad i = 1, 2, ..., n \tag{3.7}$$

so that Eq. (3.6) reduces to

$$u(x) = \lambda \sum_{i=1}^{n} C_i \, f_i(x) \tag{3.8}$$

We first find C_i as below:

Multiplying Eq. (3.8) by $g_i(x)$, (i = 1, 2, ..., n) successively and integrating over the interval [a, b], we obtain

$$\int_a^b g_1(x) u(x)\, dx = \lambda \sum_{i=1}^{n} C_i \int_a^b g_1(x) f_i(x)\, dx \tag{3.9(a)}$$

$$\int_a^b g_2(x) u(x)\, dx = \lambda \sum_{i=1}^{n} C_i \int_a^b g_2(x) f_i(x)\, dx \tag{3.9(b)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\int_a^b g_n(x) u(x)\, dx = \lambda \sum_{i=1}^{n} C_i \int_a^b g_n(x) f_i(x)\, dx \tag{3.9(n)}$$

We define $\alpha_{ji} = \int_a^b g_j(x) \cdot f_i(x) dx \quad (i, j = 1, 2, ..., n)$ (3.10)

Now using Eqs. (3.7) and (3.10), Eq. [3.9(a)] becomes

$$C_1 = \lambda \sum_{i=1}^{n} C_i \alpha_{1i}$$

or
$$C_1 = \lambda \left[C_1 \alpha_{11} + C_2 \alpha_{12} + \cdots + C_n \alpha_{1n} \right]$$

or
$$\left(1 - \lambda \alpha_{11}\right) C_1 - \lambda \alpha_{12} C_2 - \cdots - \lambda \alpha_{1n} C_n = 0$$

Similarly, we derive other equations from Eq. (3.9), and finally, obtain following simultaneous linear equations to establish C_i, which are

$$\left.\begin{array}{l}
\left(1 - \lambda \alpha_{11}\right) C_1 - \lambda \alpha_{12} C_2 - \cdots - \lambda \alpha_{1n} C_n = 0 \\
-\lambda \alpha_{21} C_1 + \left(1 - \lambda \alpha_{22}\right) C_2 - \cdots - \lambda \alpha_{2n} C_n = 0 \\
\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
-\lambda \alpha_{1n} C_1 - \lambda \alpha_{2n} C_2 - \cdots + \left(1 - \lambda \alpha_{nn}\right) C_n = 0
\end{array}\right\} \tag{3.11}$$

For this system of equations, let the determinant $D(\lambda)$ be

$$D(\lambda)=\begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & \cdots & \cdots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & \cdots & \cdots & -\lambda\alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \cdots & \cdots & (1-\lambda\alpha_{nn}) \end{vmatrix} \tag{3.12}$$

The eigenvalues of Eq. (3.3) are obtained by $D(\lambda) = 0$, and we get the maximum n eigenvalues.

3.4 ORTHOGONALITY OF TWO FUNCTIONS

Two functions $f(x)$ and $g(x)$ continuous for $a \le x \le b$, are said to be orthogonal if

$$\int_a^b f(x)g(x)dx = 0 \tag{3.13}$$

3.5 ORTHOGONALITY OF EIGENFUNCTIONS

Theorem: The eigenfunctions of a symmetric kernel corresponding to two different eigenvalues are orthogonal.

Proof and explanation: Let the homogeneous Fredholm integral equation of the second kind be

$$u(x) = \lambda\int_a^b K(x, t)\, u(t)\, dt \tag{3.14}$$

Here, kernal $K(x, t)$ is symmetric and let $\phi_0(x)$ and $\phi_1(x)$ be eigenfunctions of $K(x, t)$ corresponding to eigenvalues λ_0 and $\lambda_1(\lambda_0 \neq \lambda_1)$. Then, we are required to prove that $\phi_0(x)$ and $\phi_1(x)$ are orthogonal functions on the interval $[a, b]$, i.e.,

$$\int_a^b \phi_0(x)\, \phi_1(x)dx = 0 \tag{3.15}$$

Since $\phi_0(x)$ and $\phi_1(x)$ are eigenfunctions, by definition, these will satisfy Eq. (3.14). Thus,

$$\phi_0(x) = \lambda_0\int_a^b K(x, t)\phi_0(t)\, dt \tag{3.16}$$

and

$$\phi_1(x) = \lambda_1\int_a^b K(x,t)\phi_1(t)dt \tag{3.17}$$

Since kernel $K(x, t)$ is a symmetric function, therefore

$$K(x,t) = K(t,x) \tag{3.18}$$

Now, multiplying both sides of Eq. (3.16) by $\phi_1(x)$ and integrating with respect to x over the interval $[a, b]$, we get

$$\int_a^b \phi_0(x)\phi_1(x)dx = \lambda_0\int_a^b \phi_1(x)\left\{\int_a^b K(x,t)\phi_0(t)dt\right\}dx$$

$$= \lambda_0\int_a^b \phi_0(t)\left\{\int_a^b K(x,t)\, \phi_1(x)dx\right\}dt$$

(By changing the order of integration and now by Eq. (3.18)

$$= \lambda_0 \int_a^b \phi_0(t)\left\{\int_a^b K(t,x)\phi_1(x)dx\right\}dt \tag{3.19}$$

Now, interchanging the variables ($t \rightleftarrows x$) in Eq. (3.17), we get

$$\phi_1(t) = \lambda_1 \int_a^b K(t,x)\phi_1(x)dx$$

Using this value of $\phi_1(t)$ in Eq. (3.19), we express

$$\int_a^b \phi_0(x)\phi_1(x)\,dx = \lambda_0 \int_a^b \phi_0(t)\left\{\frac{\phi_1(t)}{\lambda_1}\right\}dt$$

or

$$\lambda_1 \int_a^b \phi_0(x)\phi_1(x)\,dx = \lambda_0 \int_a^b \phi_0(x)\phi_1(x)dx$$

By the property of definite integrals,

$$\Rightarrow \qquad (\lambda_1 - \lambda_0)\int_a^b \phi_0(x)\phi_1(x)\,dx = 0$$

Now, since $\lambda_1 \neq \lambda_0$, $(\lambda_1 - \lambda_0) \neq 0$, and hence,

$$\int_a^b \phi_0(x)\phi_1(x)\,dx = 0$$

This proves the proposition.

3.6 REAL EIGENVALUES

Theorem: The eigenvalues of a symmetric kernel are real.

Proof: We take the homogeneous Fredholm integral equation of the second kind.

$$u(x) = \lambda \int_a^b K(x,t)\,u(t)dt \tag{3.20}$$

Let $\lambda_0 = \alpha + i\beta$ be one eigenvalue and $\phi_0(x) = u + iv$ be the corresponding eigenfunction. Then,

$$\text{and} \qquad \left.\begin{aligned} \overline{\lambda}_0 &= \alpha - i\beta, \\ \overline{\phi}_0(x) &= u - iv \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \lambda_0 - \overline{\lambda}_0 &= 2i\beta, \\ \phi_0(x)\cdot\overline{\phi}_0(x) &= u^2 + v^2 \end{aligned}\right\} \tag{3.21}$$

where, a bar denotes the complex conjugate. By the definition of eigenfunction, $\phi_0(x)$ and $\overline{\phi}_0(x)$ will satisfy Eq. (3.20), and we get

$$\phi_0(x) = \lambda_0 \int_a^b K(x,t)\phi_0(t)dt \tag{3.22(a)}$$

and

$$\overline{\phi}_0(x) = \overline{\lambda}_0 \int_a^b K(x,t)\overline{\phi}_0(t)dt \tag{3.22(b)}$$

Now, multiplying both sides of Eq. [3.22(a)] by $\overline{\phi}_0(x)$ and integrating with respect to x over the interval $[a, b]$, we find

$$\int_a^b \phi_0(x)\overline{\phi}_0(x)dx = \lambda_0 \int_a^b \overline{\phi}_0(x)\left\{\int_a^b K(x,t)\phi_0(t)dt\right\}dx$$

$$\int_a^b \phi_0(x)\overline{\phi}_0(x)dx = \lambda_0 \int_a^b \phi_0(t)\left\{\int_a^b K(x,t)\overline{\phi}_0(x)dx\right\}dt$$

By changing the order of integration, and now since $K(x, t) = K(t, x)$, we have

$$\int_a^b \phi_0(x)\overline{\phi}_0(x)dx = \lambda_0 \int_a^b \phi_0(t)\left\{\int_a^b K(t,x)\overline{\phi}_0(x)dx\right\}dt \tag{3.23}$$

Now, interchanging the variables ($x \rightleftarrows t$), Eq. [3.22(b)] takes the shape as

$$\overline{\phi}_0(t) = \overline{\lambda}_0 \int_a^b K(t,x)\overline{\phi}_0(x)dx$$

and then, Eq. (3.23) is

$$\int_a^b \phi_0(x)\overline{\phi}_0(x)dx = \lambda_0 \int_a^b \phi_0(t)\left\{\frac{1}{\overline{\lambda}_0}\overline{\phi}_0(t)\right\}dt$$

or
$$\overline{\lambda}_0 \int_a^b \phi_0(x)\overline{\phi}_0(x)dx = \lambda_0 \int_a^b \phi_0(x)\overline{\phi}_0(x)dx$$

(Using the property of definite integrals)

or
$$\left(\lambda_0 - \overline{\lambda}_0\right)\int_a^b \phi_0(x)\overline{\phi}_0(x)dx = 0 \tag{3.24}$$

Using Eq. (3.21), we get

$$2i\beta\int_a^b \left(u^2 + v^2\right)dx = 0$$

Since $\phi_0(x)$ is an eigenfunction which is not zero, we infer

$$\int_a^b \left(u^2 + v^2\right)dx \neq 0$$

and thus, we conclude $\beta = 0$, which proves the proposition.

EXAMPLE 3.3: Show that the homogeneous integral equation

$$\phi(x) = \lambda\int_0^1 \left(t\sqrt{x} - x\sqrt{t}\right)\phi(t)dt$$

has no real eigenvalue and no eigenfunction.

Solution: The given equation is

$$\phi(x) = \lambda\int_0^1 \left(t\sqrt{x} - x\sqrt{t}\right)\phi(t)dt$$

or
$$\phi(x) = \lambda\sqrt{x}\int_0^1 t\phi(t)dt - \lambda x\int_0^1 \sqrt{t}\ \phi(t)dt \tag{i}$$

Let
$$c_1 = \int_0^1 t\phi(t)dt \tag{ii}$$

and
$$c_2 = \int_0^1 \sqrt{t}\ \phi(t)dt \tag{iii}$$

so that Eq. (i) is $$\phi(x) = \lambda c_1 \sqrt{x} - \lambda c_2 x$$

Thus, $$\phi(t) = \lambda c_1 \sqrt{t} - \lambda c_2 t \qquad \text{(iv)}$$

and then, by Eq. (ii), $$c_1 = \int_0^1 t\left(\lambda c_1 \sqrt{t} - \lambda c_2 t\right) dt$$

$$c_1 = \lambda c_1 \left[\frac{t^{5/2}}{5/2}\right]_0^1 - \lambda c_2 \left[\frac{t^3}{3}\right]_0^1$$

$$\Rightarrow \qquad c_1\left(1 - \frac{2\lambda}{5}\right) + \frac{\lambda}{3} c_2 = 0 \qquad \text{(v)}$$

Similarly, by Eq. (iv) and (iii)

$$c_2 = \int_0^1 \sqrt{t}\left(\lambda c_1 \sqrt{t} - \lambda c_2 t\right) dt$$

$$c_2 = \lambda c_1 \left[\frac{t^2}{2}\right]_0^1 - \lambda c_2 \left[\frac{t^{5/2}}{5/2}\right]_0^1$$

$$\Rightarrow \qquad -\frac{\lambda}{2} c_1 + \left(1 + \frac{2\lambda}{5}\right) c_2 = 0 \qquad \text{(vi)}$$

The system of Eqs. (v) and (vi) will have a non-zero solution if

$$D(\lambda) = \begin{vmatrix} 1 - \dfrac{2\lambda}{5} & \dfrac{\lambda}{3} \\ -\dfrac{\lambda}{2} & 1 + \dfrac{2\lambda}{5} \end{vmatrix} = 0$$

$$\Rightarrow \qquad 1 - \frac{4\lambda^2}{25} + \frac{\lambda^2}{6} = 0 \Rightarrow 1 + \frac{\lambda^2}{150} = 0 \Rightarrow \lambda = \pm i\sqrt{150}$$

which means that the given integral equation does not possess any real eigenvalue.

EXAMPLE 3.4: Solve the following homogeneous Fredholm integral equation of the second kind:

$$g(s) = \lambda \int_0^{2\pi} \sin(s+t) g(t) dt$$

Solution: $$g(s) = \lambda \int_0^{2\pi} \left(\sin s \cos t + \cos s \sin t\right) g(t) dt$$

$$g(s) = \lambda \sin s \int_0^{2\pi} \cos t\, g(t) dt + \lambda \cos s \int_0^{2\pi} \sin t\, g(t) dt \qquad \text{(i)}$$

Let $$c_1 = \int_0^{2\pi} \cos t\ g(t) dt \qquad \text{(ii)}$$

and $$c_2 = \int_0^{2\pi} \sin t\ g(t) dt \qquad \text{(iii)}$$

so that Eq. (i) reduces to

or
$$\left.\begin{aligned} g(s) &= \lambda \sin s \cdot c_1 + \lambda \cos s \cdot c_2 \\ g(t) &= \lambda c_1 \sin t + \lambda c_2 \cos t \end{aligned}\right\} \tag{iv}$$

Now, by Eq. (ii),

$$c_1 = \int_0^{2\pi} \cos t \cdot \left\{\lambda c_1 \sin t + \lambda c_2 \cos t\right\} dt$$

or
$$c_1 = \frac{\lambda c_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

or
$$c_1 = 0 + \frac{\lambda c_2}{2}[2\pi] \Rightarrow c_1 - \lambda \pi c_2 = 0 \tag{v}$$

Similarly, by Eqs. (iv) and (iii),

$$c_2 = \int_0^{2\pi} \sin t\{\lambda c_1 \sin t + \lambda c_2 \cos t\}\, dt$$

$$c_2 = \frac{\lambda c_1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

or
$$c_2 = \lambda c_1 \pi \Rightarrow \lambda c_1 \pi - c_2 = 0 \tag{vi}$$

For non-zero solution of this system of linear equations,

$$D(\lambda) = 0 \Rightarrow \begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0 \Rightarrow -1 + \lambda^2\pi^2 = 0$$

or
$$\lambda = \pm\frac{1}{\pi}$$

Hence, the eigenvalues are given by $\lambda_1 = \frac{1}{\pi}$ and $\lambda_2 = -\frac{1}{\pi}$.

Now, determining eigenfunction,

(a) For $\lambda = \frac{1}{\pi}$, Eqs. (v) and (vi) are $c_1 - c_2 = 0$; hence, from Eq. (iv),

$$g(s) = \frac{1}{\pi} c_1 \sin s + \frac{1}{\pi} \cos s \cdot c_1$$

$$g(s) = \frac{c_1}{\pi}(\sin s + \cos s)$$

or
$$g(s) = (\sin s + \cos s), \text{ taking } \frac{c_1}{\pi} = 1$$

(b) For $\lambda = -\frac{1}{\pi}$, Eqs. (v) and (vi) are $c_1 + c_2 = 0$,

Hence, from Eq. (iv)

$$g(s) = \frac{-1}{\pi} c_1 \sin s + \frac{-1}{\pi}(-c_1)\cos s$$

$$g(s) = \frac{-c_1}{\pi}(\sin s - \cos s)$$

or $$g(s) = (\sin s - \cos s),\ \text{taking}\ \frac{-c_1}{\pi} = 1$$

Finally, for the eigenvalue $\lambda = \frac{1}{\pi}$, the eigenfunction is $g(s) = \sin s + \cos s$ and for eigenvalue $\lambda = -\frac{1}{\pi}$, the eigenfunction is $g(s) = \sin s - \cos s$.

EXAMPLE 3.5: Find the eigenvalues and eigenfunctions of the following homogeneous integral equation:

$$\phi(x) = \lambda \int_0^{\pi} (\cos^2 x \cdot \cos 2t + \cos 3x \cdot \cos^3 t)\,\phi(t)\,dt$$

Solution: The given integral equation can be expressed as

$$\phi(x) = \lambda \cos^2 x \int_0^{x} \cos 2t \cdot \phi(t) dt + \lambda \cos 3x \int_0^{\pi} \cos^3 t \cdot \phi(t)\,dt \qquad \text{(i)}$$

Let $$c_1 = \int_0^{x} \cos 2t\, \phi(t) dt \qquad \text{(ii)}$$

and $$c_2 = \int_0^{x} \cos^3 t\, \phi(t)\,dt \qquad \text{(iii)}$$

so that Eq. (i) is

$$\phi(x) = \lambda c_1 \cos^2 x + \lambda c_2 \cos 3x$$

or $$\phi(t) = \lambda c_1 \cos^2 t + \lambda c_2 \cos 3t \qquad \text{(iv)}$$

Then, by Eq. (ii),

$$c_1 = \int_0^{\pi} \cos 2t \cdot (\lambda c_1 \cos^2 t + \lambda c_2 \cos 3t)\,dt$$

or $$c_1 \left(1 - \lambda \int_0^{x} \cos 2t \cdot \cos^2 t\,dt\right) - \lambda c_2 \int_0^{x} \cos 2t \cdot \cos 3t\,dt = 0 \qquad .$$

or $$c_1 \left[1 - \lambda \int_0^{\pi} \cos 2t \cdot \left(\frac{1 + \cos 2t}{2}\right) dt\right] - \lambda c_2 \int_0^{\pi} \frac{1}{2}(\cos 5t + \cos t)\,dt = 0$$

or $$c_1 \left[1 - \lambda \int_0^{\pi} \left\{\frac{\cos 2t}{2} + \frac{1}{4}(1 + \cos 4t)\right\} dt\right] - \lambda c_2 \left\{\frac{\sin 5t}{10} + \frac{\sin t}{2}\right\}_0^{\pi} = 0$$

or $$c_1 \left[1 - \lambda \left\{\frac{\sin 2t}{4} + \frac{t}{4} + \frac{\sin 4t}{16}\right\}_0^{\pi} - \lambda c_2 \cdot 0 = 0\right.$$

or $$c_1 \left[1 - \frac{\lambda \pi}{4}\right] - 0 \cdot c_2 = 0 \qquad \text{(v)}$$

Similarly, by Eqs. (iii) and (iv),

$$c_2 = \int_0^{\pi} \cos^3 t \cdot (\lambda c_1 \cos^2 t + \lambda c_2 \cos 3t) dt$$

We find that $\int_0^{\pi} \cos^5 t = 0$

and $$\int_0^{\pi} \cos^3 t \cdot \cos 3t \, dt = \int_0^{\pi} \frac{1}{4}(\cos 3t + 3\cos t) \cdot \cos 3t \, dt$$

$$= \frac{1}{4}\int_0^{\pi} \frac{1+\cos 6t}{2} dt + \frac{3}{4}\int_0^{\pi} \cos t \cos 3t \, dt$$

$$= \frac{1}{8}\left[t + \frac{\sin 6t}{6}\right]_0^{\pi} + \frac{3}{8}\left[\int_0^{\pi} (\cos 4t + \cos 2t)\right] dt$$

$$= \frac{\pi}{8} + \frac{3}{8}\left[\frac{\sin 4t}{4} + \frac{\sin 2t}{2}\right]_0^{\pi} = \frac{\pi}{8}$$

$\therefore$ $$c_2 = 0 \cdot c_1 + \lambda \frac{\pi}{8} c_2 \quad \text{or} \quad 0c_1 + c_2\left(1 - \frac{\lambda\pi}{8}\right) = 0 \qquad \text{(vi)}$$

For non-zero solution of system of Eqs. (v) and (vi), $D(\lambda) = 0$.

$$\begin{vmatrix} 1 - \frac{\lambda\pi}{4} & 0 \\ 0 & 1 - \frac{\lambda\pi}{8} \end{vmatrix} = 0$$

or $$\Rightarrow \left(1 - \frac{\lambda\pi}{4}\right)\left(1 - \frac{\lambda\pi}{8}\right) = 0$$

$$\Rightarrow \lambda = \frac{8}{\pi}, \frac{4}{\pi}$$

which are the eigenvalues.

Now, determining eigenfunctions,

(a) For $\lambda = 4/\pi$; by Eq. (v), $c_1 = 0 + c_2 \cdot 0 = 0$ and by Eq. (vi), $c_2\left(1 - \frac{4}{\pi} \cdot \frac{\pi}{8} = 0\right)$ $\Rightarrow c_2 = 0, c_1$ is arbitrary

Hence, by Eq. (iv),

$$\phi(x) = \lambda c_1 \cos^2 x + 0$$

or $$\phi(x) = \frac{4}{\pi} c_1 \cos^2 x$$

or $$\phi(x) = \cos^2 x \quad \left(\text{taking } \frac{4c_1}{\pi} = 1\right)$$

(b) For $\lambda = 8/\pi$; by Eq. (v), $c_1\left[1-\frac{\pi}{4}\cdot\frac{8}{\pi}\right]+0\cdot c_1 = 0$

and by Eq. (vi), $0c_1 + c_2\left(1-\frac{\pi}{8}\cdot\frac{8}{\pi}=0\right)$ $\Bigg\}\Rightarrow c_1 = 0$ and c_2 is arbitrary

Then [by Eq. (iv)] the corresponding eigenfunction is

$$\phi(x) = \frac{8}{\pi}\cdot 0\cdot\cos^2 x + \frac{8}{\pi}\cdot c_2\cos 3x$$

$$\phi(x) = \cos 3x \qquad \left(\text{taking } \frac{8c_2}{\pi} = 1\right)$$

EXERCISE 3.1

Find the eigenvalues and the corresponding eigenfunctions of the following integral equations:

1. $u(x) = \lambda\int_0^1 \sin\pi x\cdot\cos\pi t\cdot u(t)\,dt$

2. $\phi(x) = \lambda\int_0^1 (2xt - 4x^2)\phi(t)\,dt$

3. $g(s) = \lambda\int_1^2 \left(st + \frac{1}{st}\right)g(t)\,dt$

4. $\phi(x) = \lambda\int_{-1}^1 (5xt^3 + 4x^2t + 3xt)\phi(t)\,dt$

5. $g(x) = \lambda\int_0^\pi [\cos^2 x\cos 2t + \cos 3x\cos^3 t]g(t)\,dt$

Answers:

1. No eigenvalue

2. $-3, -3, (x - 2x^2)$

3. $\frac{1}{2}\left(17\pm\sqrt{265}\right), s - 2\cdot 2732\left(\frac{1}{s}\right), s + 0.4399\left(\frac{1}{s}\right)$

4. $\frac{1}{4}, \phi(x) = x^2 + \frac{3x}{2}$

5. $\frac{4}{\pi}, \frac{8}{\pi}, g_1(x) = \cos^2 x, g_2(x) = \cos 3x$

CHAPTER 4

Fredholm Integral Equations with Separable Kernels

4.1 INTRODUCTION

Like the previous chapter, this chapter is also devoted to Fredholm integral equations of the second kind. But, now the equation is not homogeneous, i.e., $F(x) \neq 0$ in general. Furthermore, the nature of kernel is separable. This chapter shows that the role of characteristic function with determinant $D(\lambda)$ is of vital importance. All interrelated possible cases are included with suitable examples.

4.2 SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND WITH SEPARABLE (OR DEGENERATE) KERNEL

We consider the following integral equation:

$$u(x) = F(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{4.1}$$

where, kernel $K(x, t)$ is separable, and therefore, we express

$$K(x,t) = \sum_{i=1}^{n} f_i(x).g_i(t) \tag{4.2}$$

which on substituting in Eq. (4.1) provides

$$u(x) = F(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} f_i(x).g_i(t) \right] u(t)dt$$

or
$$u(x) = F(x) + \lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t)u(t)dt \tag{4.3}$$

(upon interchanging the order of integration and summation)

We prescribe $\int_a^b g_i(t)u(t)dt = c_i, \qquad (i = 1, 2, ..., n)$ (4.4)

which shapes Eq. (4.3) as below:

$$u(x) = F(x) + \lambda \sum_{i=1}^{n} c_i f_i(x) \tag{4.5}$$

Now, for determining c_i's, we multiply Eq. (4.5) by $g_1(x)$, $g_2(x)$, ..., $g_n(x)$ successively and integrate over the interval $[a, b]$, and frame

$$\int_a^b g_1(x)u(x)dx = \int_a^b F(x)g_1(x)dx + \lambda \sum c_i \int_a^b g_1(x)\cdot f_i(x)dx \qquad [4.6(a)]$$

$$\int_a^b g_2(x)u(x)dx = \int_a^b F(x)g_2(x)dx + \lambda \sum c_i \int_a^b g_2(x)\cdot f_i(x)dx \qquad [4.6(b)]$$

....

....

....

$$\int_a^b g_n(x)u(x)dx = \int_a^b F(x)g_n(x)dx + \lambda \sum c_i \int_a^b g_n(x)\cdot f_i(x)dx \qquad [4.6(n)]$$

We now define

$$\alpha_{ji} = \int_a^b g_j(x) f_i(x)dx\,, \qquad (i, j = 1, 2, ..., n) \tag{4.7}$$

and

$$\beta_j = \int_a^b g_j(x)F(x)dx\,, \qquad (j = 1, 2, ..., n) \tag{4.8}$$

Now, using Eqs. (4.4), (4.7), (4.8) for [4.6(a)], we obtain

$$c_1 = \beta_1 + \lambda \sum_{i=1}^{n} c_i \alpha_{1i}$$

or

$$c_1 = \beta_1 + \lambda[c_1\alpha_{11} + c_2\alpha_{12} + \cdots + c_n\alpha_{1n}]$$

or

$$\left(1 - \lambda\alpha_{11}\right)c_1 - \lambda\alpha_{12}c_2 - \cdots - \lambda\alpha_{1n}c_n = \beta_1 \qquad [4.9(a)]$$

A similar simplification for Eqs. [4.6(b), ..., 4.6(n) will provide

$$-\lambda\alpha_{21}c_1 + \left(1 - \lambda\alpha_{22}\right)c_2 - \cdots - \lambda\alpha_{2n}c_n = \beta_2 \qquad [4.9(b)]$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$-\lambda\alpha_{n1}c_1 - \lambda\alpha_{n2}c_2 - \cdots + \left(1 - \lambda\alpha_{nn}\right)c_n = \beta_n \qquad [4.9(n)]$$

The determinant $D(\lambda)$ of this system of linear equations, i.e., from Eq. [4.9(a)] to [4.9(n)] gives

$$D(\lambda) = \begin{vmatrix} \left(1 - \lambda\alpha_{11}\right) & -\lambda\alpha_{12} & \cdots & \cdots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & \left(1 - \lambda\alpha_{22}\right) & \cdots & \cdots & -\lambda\alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \cdots & \cdots & \left(1 - \lambda\alpha_{nn}\right) \end{vmatrix}$$

$D(\lambda) = 0$ is a polynomial of utmost degree n in λ. $D(\lambda)$ does not vanish identically, since $D(0) = 1$. We consider the following three cases:

Case 1: When $F(x) = 0$, then by Eq. (4.8), $\beta_j = 0$, which makes the system [Eq. (4.9)] a homogeneous linear equation. We consider the following two situations:

(a) If $D(\lambda) \neq 0$, then we get a unique zero solution, i.e., $c_1 = c_2 = \cdots = c_n = 0$ for Eq. (4.9). So Eq. (4.1) has the solution $u(x) = 0$.
(b) If $D(\lambda) = 0$, those values of λ for which $D(\lambda) = 0$ are the eigen values and any non-zero solution of homogeneous [$\because$ $F(x) = 0$] Fredholm integral equation is known as *eigenfunction* of the integral equation.

Case 2: When $F(x) \neq 0$, but

$$\int_a^b g_j(x)F(x) = 0, \qquad j = 1, 2, ..., n,$$

i.e., $F(x)$ is orthogonal [see Eq. (3.3)] to all functions $g_j(x)$, then by Eq. (4.8), $\beta_j = 0$. ($j = 1, 2, ..., n$), and hence, the system [Eq. (4.9)] reduces to homogoneous linear equations. We find the following two situations:

(a) If $D(\lambda) \neq 0$, then the unique zero solution is $c_i = 0$ for Eq. (4.9), which provides the solution of Eq. (4.1) as $u(x) = F(x)$, [here, we use Eq. (4.5)).
(b) If $D(\lambda) = 0$, then the system [Eq. (4.9)] provides infinite non-zero solutions, and thus Eq. (4.1) has infinite non-zero solutions. The resulting solution is the sum of $F(x)$ together with the arbitrary multiples of eigenfunctions.

Case 3: When at least one of β_j is not zero, we have the following two situations:

(a) If $D(\lambda) \neq 0$, we get a unique non-zero solution of Eq, (4.9), and hence, a unique non-zero solution of Eq. (4.1) is obtained by Eq. (4.5).
(b) If $D(\lambda) = 0$, we get either no solution or infinite solutions, and thus, Eq. (4.1) has no solution or infinite solutions.

***EXAMPLE 4.1*:** Solve $g(s) = f(s) + \lambda\int_0^1 st\, g(t)dt.$

Solution: The given integral equation is

$$g(s) = f(s) + \lambda s\int_0^1 t\, g(t)dt \tag{i}$$

Let
$$c = \int_0^1 t\, g(t)dt \tag{ii}$$

so that
$$g(s) = f(s) + \lambda sc \tag{iii}$$

or
$$g(t) = f(t) + \lambda tc \tag{iv}$$

by which Eq. (ii) provides $c = \int_0^1 t[f(t) + \lambda tc]dt$

or $$c=\int_0^1 t\ f(t)dt+\lambda c\left[\frac{t^3}{3}\right]_0^1$$

or $$c=\int_0^1 t\ f(t)dt+\frac{\lambda c}{3}$$

or $$c\left(1-\frac{\lambda}{3}\right)=\int_0^1 t\ f(t)dt \Rightarrow c=\frac{3}{3-\lambda}\int_0^1 t\ f(t)dt, \lambda \neq 3$$

Then, by Eq. (iii), the solution Eq. (i) is

$$g(s)=f(s)+\frac{3\lambda s}{3-\lambda}\int_0^1 t\, f(t)dt$$

EXAMPLE 4.2: Solve $\phi(x)=\cos x+\lambda\int_0^{\pi}\sin x\,\phi(t)dt.$

Solution: The given integral equation is

$$\phi(x)=\cos x+\lambda\sin x\int_0^{\pi}\phi(t)dt \tag{i}$$

Let $$c=\int_0^{\pi}\phi(t)dt \tag{ii}$$

so that $$\phi(x)=\cos x+\lambda\sin x.c \tag{iii}$$

Hence, $$\phi(t)=\cos t+\lambda\sin t.c \tag{iv}$$

$\therefore$ $$c=\int_0^{\pi}(\cos t+\lambda c\sin t)dt$$

or $$c=[\sin t]_0^{\pi}+\lambda c[-\cos t]_0^{\pi}$$

$$=0+\lambda c[-(-1)+1]$$

$$\Rightarrow c=2\lambda c$$

$$c(1-2\lambda)=0$$

or $$\Rightarrow c=0\ \ if\ \ \lambda\neq\frac{1}{2}$$

$\therefore$ By Eq. (iii), the solution is $\phi(x)$ = cos x, provided $\lambda\neq\frac{1}{2}$.

EXAMPLE 4.3: Solve $y(x)=1+\int_0^1(1+e^{x+t}).y(t)dt.$

Solution: The given integral equation is

$$y(x)=1+\int_0^1 y(t)dt+e^x\int_0^1 e^t y(t)dt \tag{i}$$

Let $$c_1=\int_0^1 y(t)dt \tag{ii}$$

and $$c_2=\int_0^1 e^t y(t)dt \tag{iii}$$

so that Eq. (i) is $$y(x)=1+c_1+e^x c_2 \tag{iv}$$

Then, $$y(t)=1+c_1+e^t c_2 \tag{v}$$

Therefore, $$c_1 = \int_0^1 [1 + c_1 + c_2 e^t]\,dt$$

or $$c_1 = \left[t + c_1 t + c_2 e^t \right]_0^1$$

or $$c_1 = 1 + c_1 + c_2(e-1)$$

or $$c_2 = \frac{1}{1-e} \tag{vi}$$

Now, using Eq. (v),

$$c_2 = \int_0^1 e^t [1 + c_1 + c_2 e^t]\,dt$$

$$c_2 = \left[e^t + c_1 e^t + \frac{c_2}{2} e^{2t} \right]_0^1$$

$$c_2 = (e-1)(1+c_1) + \frac{c_2}{2}(e^2 - 1)$$

Putting for c_2,

$$\left(\frac{1}{1-e}\right)\left(1 - \frac{(e^2-1)}{2}\right) = (e-1)(1+c_1)$$

or $$c_1 = -\frac{(e^2 - 2e + 3)}{2(e-1)^2} \tag{vii}$$

Putting c_1 and c_2 is Eq. (iv), we get the required solution.

EXAMPLE 4.4: Show that the integral equation

$$g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi} \sin(s+t)\cdot g(t)dt$$

possesses no solution for $f(s) = s$, but it possesses infinitely many solutions when $f(s) = 1$.

Solution: The given integral equation is

$$g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi} (\sin s \cos t + \cos s \sin t).g(t)dt$$

or $$g(s) = f(s) + \frac{1}{\pi}\sin s \cdot \int_0^{2\pi} \cos t\, g(t)dt + \frac{\cos s}{\pi}\int_0^{2\pi} \sin t \cdot g(t)dt \tag{i}$$

Let $$c_1 = \int_0^{2\pi} \cos t\, g(t)dt \tag{ii}$$

and $$c_2 = \int_0^{2\pi} \sin t \cdot g(t)\,dt \tag{iii}$$

so that Eq. (i) reduces as

$$g(s) = f(s) + c_1 \frac{\sin s}{\pi} + c_2 \frac{\cos s}{\pi} \tag{iv}$$

As per the given value of $f(s)$, we take the following two cases:

Case 1: When $f(s) = s$, so that

$$g(s) = s + \frac{c_1}{\pi}\sin s + \frac{c_2}{\pi}\cos s \tag{v}$$

Hence,
$$g(t) = t + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t \tag{vi}$$

which provides by Eq. (ii)

$$c_1 = \int_0^{2\pi} (\cos t)\left[t + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t\right]dt$$

or
$$c_1 = \int_0^{2\pi} t\cos t\ dt + \frac{c_1}{\pi}\int_0^{2\pi}\sin t\cos t\, dt + \frac{c_2}{\pi}\int_0^{2\pi}\cos^2 t\ dt$$

$$c_1 = \left[t\sin t\right]_0^{2\pi} - \int_0^{2\pi}\sin t\ dt + \frac{c_1}{\pi}.\frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{2\pi} + \frac{c_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

or
$$c_1 = 0 - \left[-\cos t\right]_0^{2\pi} + 0 + \frac{c_2}{2\pi}[2\pi + 0]$$

or
$$c_1 - c_2 = 0 \tag{vii}$$

Similarly, by Eqs. (iii) and (iv),

$$c_2 = \int_0^{2\pi}\sin t\cdot\left[t + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t\right]dt$$

or
$$c_2 = \int_0^{2\pi} t\sin t\ dt + \frac{c_1}{2\pi}\int_0^{2\pi}(1 - \cos 2t)dt + \frac{c_2}{2\pi}\int_0^{2\pi}\sin 2t\ dt$$

$$c_2 = \left[-t\cos t\right]_0^{2\pi} - \int_0^{2\pi}(-\cos t)dt + \frac{c_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{c_2}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

or
$$c_1 = -2\pi + \left[\sin t\right]_0^{2\pi} + \frac{c_1}{2\pi}\left[2\pi + 0\right] + 0 \tag{viii}$$

or
$$c_1 - c_2 = 2\pi$$

Looking at Eqs. (vii) and (viii), we find that this system is inconsistent, and hence, possesses no solution.

Case 2: When f(s) = 1, then by Eqs. (v) and (vi),

$$g(s) = 1 + \frac{c_1}{\pi}\sin s + \frac{c_2}{\pi}\cos s \tag{ix}$$

Hence,
$$g(t) = 1 + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t \tag{x}$$

Then by Eq. (ii), $$c_1 = \int_0^{2\pi} \cos.t \left(1 + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t\right)dt \tag{xi}$$

which provides $c_1 = c_2$.

Again, by Eqs. (iii) and (x) ,

$$c_2 = \int_0^{2\pi} \sin t \left(1 + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t\right)dt \tag{xii}$$

which provides $c_1 = c_2$.

Thus, Eqs. (xi) and (xii) provide $c_1 = c_2 = c_0$ (arbitrary constant), and hence, by Eq. (ix), we have

$$g(s) = 1 + \frac{c_0}{\pi}(\sin s + \cos s)$$

or $$g(s) = 1 + c(\sin s + \cos s)$$

It is the required solution. Clearly, c can have arbitrary value; hence, in case when $f(s) = 1$, we find infinitely many solutions to Eq. (i).

EXAMPLE 4.5: Solve the integral equation

$$\phi(x) - \lambda\int_{-\pi}^{\pi}\left(x\cos t + t^2\sin x + \cos x \sin t\right)\phi(t)dt = x$$

Solution: The given integral equation is

$$\phi(x) = x + \lambda x\int_{-\pi}^{\pi}\cos t\cdot\phi(t)dt + \lambda\sin x\int_{-\pi}^{\pi}t^2\phi(t)dt + \lambda\cos x\int_{-\pi}^{\pi}(\sin t)\cdot\phi(t)dt \tag{i}$$

Let $$c_1 = \int_{-\pi}^{\pi}\phi(t)\cdot\cos t\,dt \tag{ii}$$

$$c_2 = \int_{-\pi}^{\pi}\phi(t)\cdot t^2 dt \tag{iii}$$

$$c_3 = \int_{-\pi}^{\pi}\phi(t)\cdot\sin t\,dt \tag{iv}$$

so that Eq. (i) can be expressed as

$$\phi(x) = x + \lambda c_1 x + \lambda c_2\sin x + \lambda c_3\cos x \tag{v}$$

$\therefore$ $$\phi(t) = t + \lambda c_1 t + \lambda c_2\sin t + \lambda c_3\cos t \tag{vi}$$

Thus $$c_1 = \int_{-\pi}^{\pi}(t + \lambda c_1 t + \lambda c_2\sin t + \lambda c_3\cos t)\cos t\,dt$$

or $$c_1 = (1+\lambda c_1)\int_{-\pi}^{\pi}t\cos t\,dt + \lambda c_2\int_{-\pi}^{\pi}\sin t\cos t\,dt + \lambda c_3\int_{-\pi}^{\pi}\cos^2 t\,dt$$

$$c_1 = 0 + 0 + 2\lambda c_3\int_0^{\pi}\frac{1}{2}(1+\cos 2t)dt$$

$$\left[\int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is an odd function}\right]$$

or
$$c_1 = \lambda c_3\left[t + \frac{\sin 2t}{2}\right]_0^{\pi} \Rightarrow c_1 - \lambda\pi c_3 = 0 \quad \text{(vii)}$$

By Eqs. (iii) and (vi),

$$c_2 = \int_{-\pi}^{\pi}[t + \lambda c_1 t + \lambda c_2 \sin t + \lambda c_3 \cos t]t^2 dt$$

or
$$c_2 = \int_{-\pi}^{\pi}(1+\lambda c_1)t^3 dt + \lambda c_2\int_{-\pi}^{\pi} t^2 \sin t dt + \lambda c_3\int_{-\pi}^{\pi} t^2 \cos t\, dt$$

$$c_2 = 0 + 0 + 2\lambda c_3\left[\left\{t^2 \sin t\right\}_0^{\pi} - \int_0^{\pi} 2t \sin t\, dt\right]$$

$$c_2 = -4\lambda c_3\left[\left\{t(-\cos t)\right\}_0^{\pi} - \int_0^{\pi}(-\cos t)dt\right]$$

$$c_2 = -4\lambda c_3[\pi + 0] \Rightarrow c_2 + 4\pi\lambda c_3 = 0 \quad \text{(viii)}$$

Similarly, by Eqs. (iv) and (vi),

$$c_3 = \int_{-\pi}^{\pi}[t + \lambda c_1 t + \lambda c_2 \sin t + \lambda c_3 \cos t]\cdot \sin t\, dt$$

$$c_3 = 2(1+\lambda c_1)\int_0^{\pi} t \sin t dt + 2\lambda c_2\int_0^{\pi} \sin^2 t dt + 0$$

$$c_3 = 2(1+\lambda c_1)\left[\left\{t(-\cos t)\right\}_0^{\pi} - \int_0^{\pi}(-\cos t)dt\right] + \left[\frac{2\lambda c_2}{2}\int_0^{\pi}(1-\cos 2t)dt\right]$$

$$c_3 = 2(1+\lambda c_1)[\pi + 0] + \lambda c_2\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

or
$$c_3 = 2\pi(1+\lambda c_1) + \lambda c_2\pi$$

or
$$2\pi\lambda c_1 + \lambda c_2\pi - c_3 = -2\pi \quad \text{(ix)}$$

Solving Eqs. (vii), (viii) and (ix), we get

$$c_1 = \frac{2\pi^2\lambda}{1+2\lambda^2\pi^2},\quad c_2 = \frac{-8\pi^2\lambda}{1+2\lambda^2\pi^2},\quad c_3 = \frac{2\pi}{1+2\lambda^2\pi^2}$$

Putting these values of c_1, c_2 and c_3 in Eq. (v), we get the required solution as

$$\phi(x) = x + \frac{2\pi\lambda}{1+2\lambda^2\pi^2}(\lambda\pi x - 4\lambda\pi \sin x + \cos x)$$

EXAMPLE 4.6: Solve the integral equation

$$g(x) = f(x) + \lambda\int_0^{2\pi} \sin(x+t)g(t)\, dt$$

and discuss all possible cases.

Solution: The given integral equation is

$$g(x) = f(x) + \lambda\int_0^{2\pi} \sin(x+t)g(t)\,dt \tag{i}$$

$$g(x) = f(x) + \lambda\int_0^{2\pi} [\sin x\cos t + \cos x\cdot\sin t]g(t)\,dt$$

$$g(x) = f(x) + \lambda\sin x\cdot\int_0^{2\pi} \cos t\cdot g(t)\,dt + \lambda\cos x\cdot\int_0^{2\pi} \sin t\cdot g(t)\,dt$$

or $$g(x) = f(x) + (\lambda\sin x)c_1 + (\lambda\cos x)c_2 \tag{ii}$$

where, $$c_1 = \int_0^{2\pi} \cos t\, g(t)\,dt \tag{iii}$$

and $$c_2 = \int_0^{2\pi} \sin t\cdot g(t)\,dt \tag{iv}$$

Now, from Eq. (ii),

$$g(t) = f(t) + (\lambda\sin t)c_1 + (\lambda\cos t)c_2 \tag{v}$$

Putting Eq. (v) in Eq. (iii), we have

$$c_1 = \int_0^{2\pi} \cos t[f(t) + (\lambda\sin t)c_1 + (\lambda\cos t)c_2]\,dt$$

$$\Rightarrow \quad c_1 = \int_0^{2\pi} \cos t\cdot f(t)\,dt + \lambda c_1\int_0^{2\pi} \frac{\sin 2t}{2}\,dt + \lambda c_2\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\,dt$$

$$\Rightarrow \quad c_1 = \int_0^{2\pi} \cos t\cdot f(t)\,dt + 0 + \lambda c_2\cdot\pi$$

$$\Rightarrow \quad c_1 - \lambda\pi c_2 = \int_0^{2\pi} \cos t\cdot f(t)\,dt \tag{vi}$$

Similarly, by Eqs. (v) and (iv),

$$-\lambda\pi c_1 + c_2 = \int_0^{2\pi} \sin t\cdot f(t)\,dt \tag{vii}$$

From Eq. (vi) and (vii),

the corresponding $$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = (1-\lambda^2\pi^2) \tag{viii}$$

$$\therefore \quad D(\lambda) = 0 \Rightarrow \lambda = \pm\frac{1}{\pi} \tag{ix}$$

Case 1: When $f(x) = 0$, then Eq. (i) reduces to

$$g(x) = \lambda\int_0^{2\pi} \sin(x+t)\cdot g(t)dt$$

(a) If $D(\lambda) \neq 0$, then we shall have a unique zero solution, i.e., $c_1 = 0 = c_2$. So, Eq. (i) has the solution $g(x) = 0$.

(b) If $D(\lambda) = 0$ (case of non-trivial solution), i.e., if $\lambda = \pm\frac{1}{\pi}$, then the eigenvalues are $\lambda = \pm\frac{1}{\pi}$.

We now find eigenfunctions corresponding to $\lambda = \frac{1}{\pi}, \frac{-1}{\pi}$.

For $\lambda = \frac{1}{\pi}$, using Eq. (vi) and (vii) with $f(x) = 0$, these equations give $c_1 - c_2 = 0$, and therefore, Eq. (ii) becomes

$$g(x) = \frac{c_1}{\pi}(\sin x + \cos x) = A(\sin x + \cos x)$$

where, $A = \frac{c_1}{\pi}$ is an arbitrary constant.

Thus, $(\sin x + \cos x)$ or any non-zero constant multiple of $(\sin x + \cos x)$ will be the eigenfunction corresponding to eigenvalue $\lambda = \frac{1}{\pi}$.

Similarly, corresponding to $\lambda = \frac{-1}{\pi}$, Eq. (vi) and (vii) with $f(x) = 0$ give $c_1 + c_2 = 0$, and therefore, Eq. (ii) becomes

$$g(x) = \frac{c_2}{\pi}(\sin x - \cos x) = B(\sin x - \cos x)$$

where, $B = \frac{c_2}{\pi}$ is an arbitrary constant.

Thus, $(\sin x - \cos x)$ or any non-zero constant multiple of $(\sin x - \cos x)$ will be the eigenfunction corresponding to eigenvalue $\lambda = -\frac{1}{\pi}$.

Case 2: When $f(x) \neq 0$, but

$$\int_0^{2\pi} \cos t \cdot f(t)\, dt = 0 \text{ and } \int_0^{2\pi} \sin t \cdot f(t) = 0$$

i.e., $f(t)$ is orthogonal to $\cos t$ and $\sin t$, then Eqs. (vi) and (vii) provide homogeneous linear equations. We have the following two situations:

(a) If $D(\lambda) \neq 0$, Eqs. (vi) and (vii) provide $c_1 = 0 = c_2$. So, the solution of Eq. (i) is $g(x) = f(x)$.

(b) If $D(\lambda) = 0$, then $\lambda = \pm\frac{1}{\pi}$ and as found in Case 1, we have infinite non-zero solutions. The resulting solution of (1) is $g(x) = f(x)$ + the arbitrary multiples of $(\sin x + \cos x)$ if $\lambda = \frac{1}{\pi}$, and $(\sin x - \cos x)$ if $\lambda = -\frac{1}{\pi}$.

Case 3: When at least one of the $\int_0^{2\pi} \cos t \cdot f(t)dt \neq 0$ or $\int_0^{2\pi} \sin t \cdot f(t)dt \neq 0$ we have the following two situations:

(a) If $D(\lambda) \neq 0$, we will have a unique non-zero solution of Eq. (i).

(b) If $D(\lambda) = 0$ and $\lambda = \frac{1}{\pi}$, then Eq. (vi) and (vii) become

$$\left.\begin{aligned} c_1 - c_2 &= \int_0^{2\pi} \cos t \cdot f(t)\, dt \\ c_1 - c_2 &= -\int_0^{2\pi} \sin t \cdot f(t) dt \end{aligned}\right\} \qquad \begin{aligned}&[\text{x(a)}]\\&[\text{x(b)}]\end{aligned}$$

When $\lambda = -1/\pi$, then Eqs. (vi) and (vii) provide

$$\left.\begin{aligned} c_1 + c_2 &= \int_0^{2\pi} \cos t \cdot f(t)\, dt \\ c_1 + c_2 &= \int_0^{2\pi} \sin t \cdot f(t) dt \end{aligned}\right\} \qquad \begin{aligned}&[\text{xi(a)}]\\&[\text{xi(b)}]\end{aligned}$$

Now, Eqs. [x(a)] and [x(b)] are incompatible unless the function $f(t)$ satisfies the condition.

$$\int_0^{2\pi} \cos t \cdot f(t)\, dt = -\int_0^{2\pi} \sin t \cdot f(t)\, dt$$

or $$\int_0^{2\pi} (\cos t + \sin t) \cdot f(t)\, dt = 0 \tag{xii}$$

Similarly, Eqs. [xi(a)] and [xi(b)] are incompatible unless the function $f(t)$ satisfies the condition

$$\int_0^{2\pi} (\sin t - \cos t) \cdot f(t)\, dt = 0 \tag{xiii}$$

When Eqs. (xii) and (xiii) are satisfied, the Eqs. (x) and (xi) become redundant and we have infinitely many solutions.

Another case: When $\lambda = \dfrac{1}{\pi}$ and Eq. (xii) is satisfied, then by Eq. (x)

$$c_1 = c_2 + \int_0^{2\pi} \cos t \cdot f(t)\, dt$$

Now, putting it in Eq. (ii).

$$g(x) = f(x) + \frac{c_2 \sin x}{\pi} + \frac{\sin x}{\pi} \int_0^{2\pi} \cos t \cdot f(t)\, dt + \frac{c_2}{\pi} \cos x$$

$$g(x) = f(x) + \frac{\sin x}{\pi} \int_0^{2\pi} \cos t \cdot f(t)\, dt + A(\cos x + \sin x) \tag{xiv}$$

where, $A = c_2/\pi$ is an arbitrary constant.

Thus, if $\lambda = 1/\pi$ and $\int_0^{2\pi} (\cos x + \sin x) f(x)\, dx = 0$, Eq (i) possesses infinitely many solutions.

Similarly, when $\lambda = -1/\pi$, and Eq. (xiii) is satisfied, then by Eq. (xi),

$$c_1 = -c_2 + \int_0^{2\pi} \cos t \cdot f(t)\, dt$$

Putting it in Eq. (ii), we get

$$g(x) = f(x) - \frac{\sin x}{x} \int_0^{2\pi} \cos t \cdot f(t)\, dt + B(\sin x - \cos x) \tag{xv}$$

Thus, if $\lambda = -1/\pi$ and $\int_0^{2\pi} (\sin x - \cos x) f(x)\, dx = 0$, Eq. (i) possesses infinitely many solutions.

EXAMPLE 4.7: Solve the following integral equation and discuss its all possible cases:

$$\phi(x) = F(x) + \lambda \int_0^1 (1 - 3xt)\phi(t) dt$$

Solution:

The given integral equation is $\phi(x) = F(x) + \lambda \int_0^1 (1 - 3xt)\, \phi(t) dt$ (i)

It may be expressed as

$$\phi(x) = F(x) + \lambda \int_0^1 \phi(t)\, dt - 3x\lambda \int_0^1 t\, \phi(t) dt \tag{ii}$$

Let
$$c_1 = \int_0^1 \phi(t) dt \tag{iii}$$

and
$$c_2 = \int_0^1 t\, \phi(t) dt \tag{iv}$$

so that Eq. (ii) shapes as below:

$$\phi(x) = F(x) + \lambda c_1 - 3x\lambda c_2 \tag{v}$$

so that
$$\phi(t) = F(t) + c_1\lambda - 3\lambda c_2 t \tag{vi}$$

Thus,
$$c_1 = \int_0^1 [F(t) + c_1\lambda - 3\lambda c_2 t] dt$$

or
$$c_1 = \int_0^1 F(t) dt + c_1\lambda - 3\lambda c_2 \cdot \frac{1}{2} \tag{vii}$$

Similarly, by Eqs. (iv) and (vi),

$$c_2 = \int_0^1 t[F(t) + c_1\lambda - 3\lambda c_2 t] dt$$

or
$$c_2 = \int_0^1 tF(t) dt + c_1\lambda \cdot \frac{1}{2} - 3c_2\lambda \cdot \frac{1}{3} \tag{viii}$$

Equations (vii) and (viii) may be expressed as

$$(1 - \lambda)c_1 + \frac{3\lambda}{2} c_2 = \int_0^1 F(t) dt = \int_0^1 F(x) dx \tag{ix}$$

and
$$-\frac{\lambda}{2} c_1 + (1 + \lambda)c_2 = \int_0^1 t\, F(t) dt = \int_0^1 x\, F(x) dx \tag{x}$$

The determinant

$$D(\lambda) = \begin{vmatrix} 1 - \lambda & 3\lambda/2 \\ -\lambda/2 & 1 + \lambda \end{vmatrix} = \frac{1}{4}(4 - \lambda^2) \tag{xi}$$

Case 1: A unique solution of the system of equations i.e., Eqs. (ix) and (x) will exist if and only if $D(\lambda) \neq 0 \Rightarrow \lambda \neq \pm 2$. As discussed in Section 4.2

(Case 3), the values of c_1 and c_2 can be determined by solving this system, which in turn, after putting in Eq. (v) gives the solution of integral equation, i.e., Eq. (ii) Particularly, if $F(x) = 0$ and $\lambda \neq \pm 2$, we derive $c_1 = 0 = c_2$, which leads to trivial solution $\phi(x) = 0$. Clearly, the numbers $\lambda = \pm 2$ are the eigenvalues for the problem.

If $\lambda = 2$, Eqs. (ix) and (x) shape as

and
$$\left.\begin{aligned} -c_1 + 3c_2 &= \int_0^1 F(x)dx \\ -c_1 + 3c_2 &= \int_0^1 xF(x)dx \end{aligned}\right\} \Rightarrow \int_0^1 (1-x)F(x)dx = 0$$

This means that for $\lambda = 2$, the system of equations becomes incompatible (possessing no solution) unless $\int_0^1 (1-x)F(x) = 0$. If this is true, then we get same set from both the equations, which means one relation in c_1 and c_2, and hence, infinitely many solutions will exist.

If $\lambda = -2$, a similar argument follows.

Case 2: Let $F(x) = 0$, then Eq. (i) is homogeneous integral equation;

$$\phi(x) = \lambda \int_0^1 (1-3xt)\,\phi(t)\,dt \tag{xii}$$

If $\lambda \neq \pm 2$, then Eq. (i) has the trivial solution $\phi(x) = 0$, as mentioned in Case 1.

For non-trivial solution $\lambda = \pm 2$ are the eigenvalues, we find eigen- function for $\lambda = 2$ first. Eqs. (ix) and (x) for $F(x) = 0$ reduce to

$$c_1 = 3c_2$$

and then, by Eq. (v),

$$\phi(x) = 2(3c_2 - 3xc_2) = A(1 - x)$$

where, $A = 6c_2$ is an arbitrary constant.

Thus, in this case, for $\lambda = 2$, the eigenfunction is $A(1 - x)$.

Similarly, for $\lambda = -2$, Eq. (ix) and (x) provide $c_1 = c_2$.

Again, by Eq. (v),

$$\phi(x) = -2c_1(1-3x) = B(1-3x)$$

where $B = -2c_1$ is an arbitrary constant.

Clearly, in this case, when $\lambda = -2$, the eigenfunction is $B(1 - 3x)$.

Case 3: When $F(x) \neq 0$, then the given integral equation, i.e., Eq. (i) is non-homogeneous. We consider the following three situations.

(a) When $\lambda \neq \pm 2$: This situation has already been dealt in Case 1.

(b) When $\lambda = 2$: The compatibility is there if $F(x)$ is orthogonal to $(1 - x)$. We refer back to Case 1 [if $\lambda = 2$, and $\int_0^1 (1-x)F(x)\,dx = 0$] and we have found

$$c_1 = 3c_2 - \int_0^1 F(x)\,dx$$

which by Eq. (v) gives

$$\phi(x) = F(x) + \lambda[3c_2 - \int_0^1 F(x)\,dx] - 3xc_2\lambda$$

or $$\phi(x) = F(x) - 2\int_0^1 F(x)\,dx + A(1-x) \qquad \text{(xiii)}$$

where, $A = 6c_2$

Finally, if $\lambda = 2$ and $\int_0^1 (1-x)F(x) = 0$, Eq. (xiii) gives infinitely many solutions.

(c) When $\lambda = -2$: As before, the compatibility is there if $F(x)$ is orthogonal with $(1 - 3x)$.

i.e., $\int_0^1 (1-3x)F(x)\,dx = 0$, and we find

$$c_1 = c_2 + \frac{1}{3}\int_0^1 F(x)\,dx$$

which by Eq. (v), gives

$$\phi(x) = F(x) + \lambda\left[c_2 + \frac{1}{3}\int_0^1 F(x)\,dx\right] - 3xc_2\lambda$$

or $$\phi(x) = F(x) - \frac{2}{3}\int_0^1 F(x)\,dx + B(1-3x) \qquad \text{(xiv)}$$

where, $$B = -2c_2 = -2c_1$$

Finally, if $\lambda = -2$ and $\int_0^1 (1-3x)F(x) = 0$, Eq. (xiv) provides infinitely many solutions.

EXERCISE 4.1

Solve the following integral equations.

1. $u(x) = e^x + \lambda\int_0^1 2e^x e^t \cdot u(t)\,dt$

2. $\phi(x) = 2x - \pi + 4\int_0^{\pi/2} \sin^2 x \;\; \phi(\xi)\,d\xi$

3. $g(s) = s + \lambda\int_0^1 st(s+t)g(t)\,dt$

4. $\phi(x) = (1+x^2) + \int_{-1}^1 (xt + x^2t^2)\phi(t)\,dt$

5. $\phi(x) = \cos + \lambda\int_0^{\pi} \sin(x-t)\;\phi(t)\,dt$

6. $g(s) = f(s) + \lambda\int_{-1}^{1} st(1+st)g(t)\,dt$. Find its resolvent kernel also.
7. Invert (or solve) the integral equation:

$$g(s) = f(s) + \lambda\int_{0}^{2\pi} (\sin s \cos t)\, g(t)\,dt$$

8. Solve $g(s) = x + \lambda\int_{0}^{1} (xt^2 + x^2 t)g(t)\,dt$
9. Solve $g(x) = 1 + \lambda\int_{0}^{\pi/2} \cos(x-t)g(t)\,dt$, and find its eigenvalues.

Answers

1. $u(x) = \dfrac{e^x}{1-\lambda(e^2-1)}$, $\lambda \neq 1/(e^2-1)$
2. $\phi(x) = 2x - \pi + (\pi^2 \sin^2 x)/(\pi - 1)$
3. $g(s) = [(240 - 60\lambda) + 80\lambda s^2]/(240 - 120\lambda - \lambda^2)$
4. $\phi(x) = 1 + 6x + (25/9)x^2$ or $\phi(x) = (1+x)^2 + 4x + (16/9)x^2$
5. $\phi(x) = \{4\cos x + 2\pi\lambda \sin x\}/(4 + \pi^2\lambda^2)$
6. $g(s) = f(s) + \lambda\int_{-1}^{1}\left\{\dfrac{3st}{3-2\lambda} + \dfrac{5s^2t^2}{5-2\lambda}\right\} f(t)\,dt$

   resolvent kernel $R(s, t; \lambda) = \dfrac{3st}{3-2\lambda} + \dfrac{5s^2t^2}{5-2\lambda}$
7. $g(s) = \{2/(2-\lambda)\} \sin s$, $\lambda \neq 2$
8. $g(x) = \dfrac{(240-60\lambda)x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$
9. $g(x) = 1 + \dfrac{\lambda(\cos x + \sin x)}{1 - \pi(\pi+2)/4}$, $\lambda = 4/(\pi \pm 2)$

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CHAPTER 5

Integral Equations with Symmetric Kernels

5.1 INTRODUCTION

In the previous chapters, Fredholm integral equations of the second kind have been considered for any given kernal $K(x, t)$ by having the eigenvalues and corresponding eigenfunctions. In this chapter, the same equation is the main motive, but now, the kernel is symmetric.

5.2 SYMMETRIC KERNAL

A kernel $K(x, t)$ is said to be symmetric (also complex symmetric or Hermitian) if

$$K(x,t)=\bar{K}(t,x) \tag{5.1}$$

where, the bar denotes the complex conjugate. If the kernel is real, the symmetry reduces to equality.

$$K(x,t)=K(t,x) \tag{5.2}$$

Theorem: If a kernel is symmetric, then all its iterated kernels are also symmetric.

Proof: Let the kernel $K(x, t)$ be symmetric. Then, by defintion,

$$K(x,t)=\bar{K}(t,x) \tag{5.3}$$

By definition, the iterated kernels $K_n(x,t), n \in I^+$ are defined as

$$K_1(x,t)=K(x,t) \qquad [5.4(a)]$$

$$K_n(x,t)=\int_a^b K(x,z).K_{n-1}(z,t)\,dz, \quad n = 2,3, \ldots \qquad [5.4(b)]$$

Also, $$K_n(x,t)=\int_a^b K_{n-1}(x,z).K(z,t)\,dz, \quad n = 2, 3, \ldots \qquad [5.4(c)]$$

We shall use the principle of mathematical induction to prove the required results.

By Eq. [(5.4(b)],

$$K_2(x,t) = \int_a^b K(x,z).K_1(z,t)\,dz$$

Now, by Eq. [(5.4(a)],

$$K_2(x,t) = \int_a^b K(x,z).K(z,t)\,dz$$

and By Eq. (5.3)

$$K_2(x,t) = \int_a^b \bar{K}(z,x).\bar{K}(t,z)\,dz$$

$$K_2(x,t) = \int_a^b \bar{K}(t,z).\bar{K}(z,x)\,dz$$

By Eq. [(5.4(a)],

$$K_2(x,t) = \int_a^b \bar{K}(t,z)\cdot\bar{K}_1(z,x)\,dz$$

Thus, $$K_2(x,t) = \bar{K}_2(t,x)$$

which shows that $K_2(x, t)$ is symmetric by definition, and the required result is true for $n = 1, 2$.

Let $K_n(x, t)$ be symmetric for $n = m$. Then, by definition,

$$K_m(x,t) = \bar{K}_m(t,x) \qquad [5.4(d)]$$

We shall show that $K_n(x, t)$ is also symmetric for $n = m + 1$, i.e.,

$$K_{m+1}(x,t) = \bar{K}_{m+1}(t,x) \qquad [5.4(e)]$$

Now, by Eq. [5.4(b)],

$$K_{m+1}(x,t) = \int_a^b K(x,z)\cdot K_m(z,t)\,dz$$

Now, using Eqs. (5.3) and [5.4(d)],

$$K_{m+1}(x,t) = \int_a^b \bar{K}(z,x)\cdot\bar{K}_m(t,z)\,dz$$

$$K_{m+1}(x,t) = \int_a^b \bar{K}_m(t,z)\cdot\bar{K}(z,x)\,dz$$

which upon using Eq. [5.4(c)] becomes

$K_{m+1}(x,t) = \bar{K}_{m+1}(t,x)$, which is the R.H.S. of Eq. [5.4(e)].

Thus, the iterated kernel $K_n(x, t)$ is symmetric for $n = 1, 2$; and it is also symmetric for $n = m + 1$ whenever it is true for $n = m$. Hence, by mathematical induction, $K_n(x, t)$ is symmetric for $n = 1, 2, \ldots$.

5.3 REGULARITY CONDITION

In our study, the functions are either continuous or integrable or square

integrable. When an integral sign is used, it is to be taken as Lebesgue integral. Moreover, we know that a function which is Riemann integrable, it is also Lebesgue integrable. By a square integrable function $f(x)$, we mean that

$$\int_a^b |f(x)|^2 \, dx < \infty$$

For example, $\int_0^{\pi/2} \sec^2 x \, dx \to \infty$ and it (sec x) is not square integrable.

A square integrable function $f(x)$ is also called L_2 function. A kernel $K(x,t)$ is an L_2 function if it satisfies the following three conditions:

1. $\int_a^b \int_a^b |K(x,t)|^2 \, dx \, dt < \infty, \quad \forall x \in [a,b], \forall t \in [a,b]$
2. $\int_a^b |K(x,t)|^2 \, dt < \infty, \quad \forall x \in [a,b]$, and
3. $\int_a^b |K(x,t)|^2 \, dx < \infty, \quad \forall t \in [a,b]$

5.4 INNER OR SCALAR PRODUCT OF TWO FUNCTIONS

The inner or scalar product of two complex functions ϕ and ψ of real variable $x, a \le x \le b$ is denoted by (ϕ, ψ) and is defined as $(\phi, \psi) = \int_a^b \phi(x) \cdot \overline{\psi}(x) \, dx$, where the bar denotes the complex conjugate.

The norm of a function $\phi(x)$ is given by the following relation:

$$\| \phi(x) \| = \left[\int_a^b \phi(x) . \overline{\phi}(x) \, dx \right]^{1/2} = \left[\int_a^b |(\phi)x|^2 \, dx \right]^{1/2}$$

A function $\phi(x)$ is said to be normalised if $\|\phi(x)\| = 1$. It follows that a non-null function (whose norm is not zero) can always be normalised by dividing it by its norm.

Note: For the Fredholm linear operator K,

$$K\phi = \int_a^b K(x,t) \cdot \phi(t) \, dt$$

The operator adjoint to K is

$$\overline{K}\psi = \int_a^b \overline{K}(t,x) \cdot \psi(t) \, dt$$

The two operators $K\phi$ and $\overline{K}\psi$ are connected as follows:

$$(K\phi, \psi) = (\phi, \overline{K}\psi)$$

For a symmetric kernel, it reduces to $(K\phi, \psi) = (\phi, K\psi)$, which means that a symmetric kernel is self-adjoint. Now, since the permutation of factors in a scalar product is equivalent to taking the complex conjugate, so $(\phi, K\phi) = \overline{(K\phi, \phi)}$. Combining this with $(K\phi, \psi) = (\phi, K\psi)$, we find that for a symmetric kernel, the inner product $(K\phi, \phi)$ is always real. The converse of this is also true.

5.5 ORTHOGONAL SYSTEM OF FUNCTIONS

A finite or an infinite set $\{\phi_k(x)\}$ defined on an interval $a \le x \le b$ is said to be an orthogonal set if $(\phi_i,\phi_j) = 0$ or $\int_a^b \phi_i(x).\phi_j(x)\,dx = 0, i \neq j$. For instance, $\int_0^{2\pi} \sin x.\cos x\,dx = 0$; so $\sin x$ and $\cos x$ are orthogonal functions for $0 \le x \le 2\pi$.

If none of the elements of this set is a zero vector, then it is called *proper orthogonal set*. The set $\{\phi_i(x)\}$ is orthonormal if

$$(\phi_i,\phi_j) = \int_a^b \phi_i(x)\ \phi_j(x) dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Any function $\phi(x)$ for which $\|\phi(x)\| = 1$ is said to be normalised.

Given a finite or an infinite (denumerable) independent set of functions $\{\psi_1, \psi_2, ..., \psi_k ...\}$ we can construct an orthonormal set $\{\phi_1, \phi_2, ..., \phi_k ...\}$. We have an important theorem (without proof) relating such functions.

Riesz–Fisher Theorem: If $\{\phi_i(x)\}$ is a given orthogonal system of functions in L_2 and $\{\alpha_i\}$ is a given sequence of complex numbers such that the series $\sum_{i=1}^{\infty} |\alpha_i|^2$ converges, then there exists a unique function $f(x)$ for which α_i are the Fourier coefficients with respect to the orthonormal system $\{\phi_i(x)\}$ and to which the Fourier series converges in the mean, i.e.,

$$\| f(x) - \sum_{i=1}^{\infty} \alpha_i.\phi_i(x) \| \to 0 \text{ as } n \to \infty$$

5.6 FUNDAMENTAL PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS OF SYMMETRIC KERNELS

We consider the following symmetric integral equation:

$$\lambda \int_a^b K(x,t).g(t)\,dt = f(x) \text{ or } \lambda K g = f;\ K(x,t) = \bar{K}(t,x)$$

Property 1: The eigenvalues of a symmetric kernel are real.

Proof: Let λ and $\phi(x)$ be an eigenvalue and a corresponding eigenfunction of kernal $K(x, t)$. Then, by definition of eigenfunction,

$$\phi(x) = \lambda \int_a^b K(x,t)\phi(t)\,dt$$

or,
$$\phi(x) - \lambda K\phi(x) = 0$$

Multiplying by $\bar{\phi}(x)$ and integrating with respect to x from a to b, we obtain

$$\| \phi(x) \|^2 - \lambda (K\phi, \phi) = 0$$

or
$$\lambda = \| \phi(x) \|^2 / (K\phi, \phi)$$

Since both N^r and D^r for R.H.S. are real, hence λ is real.

Property 2: The eigenfunctions of symmetric kernel, corresponding to different eigenvalues are orthogonal.

Proof: Let ϕ_1 and ϕ_2 be eigenfunctions corresponing to eigenvalues λ_1 and λ_2, respectively, where $\lambda_1 \neq \lambda_2$. Then by definition, we have

$$\phi_1 - \lambda_1 K\phi_1 = 0 \tag{5.5}$$

$$\phi_2 - \lambda_2 K\phi_2 = 0 \tag{5.6}$$

Since λ_2 is real, Eq. (5.6) may written as

$$\bar{\phi}_2 - \lambda_2 \bar{K}\bar{\phi}_2 = 0 \tag{5.7}$$

Multiplying Eq. (5.5) by $\lambda_2 \bar{\phi}_2$ and Eq. (5.7) by $\lambda_1\phi_1$, subtracting and integrating, we get $\left[\text{recollect}\,(\phi,\psi) = \int_a^b \phi(x)\bar{\psi}(x)\,dx\right]$,

$$(\lambda_2 - \lambda_1)(\phi_1,\phi_2) = \lambda_1\lambda_2\left[\int_a^b\int_a^b \bar{\phi}_2(x)\bar{K}(x,t)\phi_1(t)\,dt\,dx - \int_a^b\int_a^b \phi_1(x)\bar{K}(x,t)\bar{\phi}_2(t)\,dt\,dx\right] \tag{5.8}$$

Since kernel $K(x, t)$ is symmetric, we have

$$K(x,t) = \bar{K}(t,x) \tag{5.9}$$

Using Eq. (5.9), we find that R.H.S. of Eq. (5.8) vanishes, and so, we get

$$(\lambda_2 - \lambda_1)(\phi_1,\phi_2) = 0 \tag{5.10}$$

Now, since $\lambda_2 \neq \lambda_1$, Eq. (5.10) reduces to $(\phi_1,\phi_1) = 0$,
which means ϕ_1 and ϕ_2 are orthogonal.

Property 3: The multiplicity of any non-zero eigenvalue is finite for every symmetric kernel for which $\int_a^b\int_a^b |K(x,t)|^2 dx\,dt$ is finite.

Property 4: The eigenvalues of a symmetric L_2 kernel forms a finite or an infinite sequence $\{\lambda_n\}$ with no finite limit point.

Property 5: The set of eigenvalues of the second iterated kernel coincide with the set of squares of the eigenvalues of the given kernel.

Property 6: The sequence of eigenfunctions of a symmetric kernel can be made orthonormal.

5.7 HILBERT–SCHMIDT THEOREM

If $f(x)$ can be written in the form

$$f(x) = \int_a^b K(x,t)\,h(t)\,dt \tag{5.11}$$

where $K(x, t)$ is a symmetric–kernel and $h(t)$ is an L_2–function, then $f(x)$ can

be expanded in an absolutely and uniformly convergent Fourier series with respect to the orthonormal system of eigenfunctions $\phi_1(x), \phi_2(x), \ldots \phi_n(x), \ldots$ of the kernel $K(x, t)$.

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \tag{5.12}$$

where,

$$f_n = (f, \phi_n) \tag{5.13}$$

The Fourier coefficients f_n of the functions $f(x)$ are related to the Fourier coefficients h_n of the functions $h(x)$ by the following relations:

$$f_n = h_n/\lambda_n \tag{5.14}$$

and

$$h_n = (h, \phi_n) \tag{5.15}$$

where λ_n are the eigenvalues of kernel $K(x, t)$.

Proof: Let $K(x, t)$ be a non-null, symmetric kernel which has a finite or an infinite number of (real and non-zero) eigenvalues, ordering them in the sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \tag{5.16}$$

in such a way that each eigenvalue is repeated as many times as its multiplicity. We further agree to denumerate these eigenvalues in the order that corresponds to their absolute value, i.e.,

$$0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n| \le |\lambda_{n+1}| \le \cdots$$

Let

$$\phi_1(x), \phi_2(x), \cdots, \phi_n(x) \cdots \tag{5.17}$$

be the sequence of eigenfunctions corresponding to the eigenvalues given by the sequence shown in Eq. (5.16) and arranged in such a way that they are no longer repeated and are linearly independent in each group corresponding to the same eigenvalue.

Thus, to each eigenvalue λ_k in Eq. (5.16), there corresponds just one eigenfunction $\phi_k(x)$ in Eq. (5.17). Further, we suppose eigenfunctions $\phi_k(x)$ in Eq. (5.12) have been orthonormalised.

Now, the Fourier coefficients f_n of the function $f(x)$ with respect to the orthonormal system $\{\phi_n(x)\}$ are [given by Eq. (5.13)]

$$f_n = (f, \phi_n) = (Kh, \phi_n) = (h, K\phi_n)$$

$f_n = \dfrac{1}{\lambda_n}(h\phi_n) = \dfrac{h_n}{\lambda_n}$, using the self-adjoint property of the operator and the relation $\lambda_n K\phi_n = \phi_n$.

Hence, the Fourier series for $f(x)$ is given by Eq. (5.12) and is expressed as

$$f(x) \sim \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n} \phi_n(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \tag{5.18}$$

We now estimate the remainder term of the series Eq. (5.18), as shown below:

$$\left|\sum_{k=n+1}^{n+p} h_k \frac{\phi_k(x)}{\lambda_k}\right|^2 \le \left(\sum_{k=n+1}^{h+p} h_k^2\right)\left(\sum_{k=n+1}^{h+p} \frac{|\phi_k(x)|^2}{\lambda_k^2}\right)$$

$$\le \left(\sum_{k=n+1}^{h+p} h_k^2\right)\left(\sum_{k=1}^{\infty} \frac{\phi_k^2(x)}{\lambda_k^2}\right) \tag{5.19}$$

Using Bessel's inequality,

$$\sum_{n=1}^{\infty} \frac{|\phi_n(x)|^2}{\lambda_n^2} \le \int_a^b |K(x\,t)|^2\, dt \le C_1^2$$

We find that the above series is bounded. Moreover, since $h(x)$ is an L_2 function, it follows that the series $\sum_{k=1}^{\infty} h_k^2$ is convergent and the partial sum $\sum_{k=n+1}^{n+P} h_k^2$ can be made arbitrarily small. Hence, Eq. (5.18) converges absolutely and uniformly.

We now proceed to show that Eq. (5.18) converges to $f(x)$ in the mean. For this purpose, let us denote its partial sum as

$$\psi_n(x) = \sum_{m=1}^{n} \frac{h_m}{\lambda_m}\, \phi_m(x) \tag{5.20}$$

and estimate the value of $\| f(x) - \psi_n(x) \|$.

Now, $f(x) - \psi_n(x) = Kh - \sum_{m=1}^{n} \frac{h_m}{\lambda_m}\, \phi_m(x)$ [For h_m, refer to Eq. (5.15)]

$$= Kh - \sum_{m=1}^{n} \frac{(h,\, \phi_m)}{\lambda_m}\, \phi_m(x) = K^{(m+1)}h \tag{5.21}$$

where $K^{(n+1)}$ is the truncated kernel. From Eq. (5.21), we have

$$\| f(x) - \psi_n(x) \|^2 = \| K^{(n+1)}h \|^2 = (K^{(n+1)}h, K^{(n+1)}h)$$

$$= (h, K^{(n+1)}K^{(n+1)}h) = (h, K_2{}^{(n+1)}h) \tag{5.22}$$

where we have used the self-adjointness property of kernel $K^{(n+1)}$ and also the relation $K^{(n+1)}K^{(n+1)}h = K_2{}^{(n+1)}$.

We know that the set of eigenvalues of the second iterated kernel coincides with the set of squares of the eigenvalues of the given kernel. Using this property, we see that the least eigenvalue of kernel $K_2^{(n+1)}$ is equal to

$$\lambda_n{}^2 + 1\text{. Again, } \frac{1}{\lambda_{n+1}^2} = \max\left[\frac{(h, K_2^{(n+1)}h)}{(h,\, h)}\right] \tag{5.23}$$

where we have omitted the modulus sign from the scalar product $(h, K_2{}^{(n+1)}h)$ because it is a positive quantity.

Combining Eq. (5.22) and (5.23), we find

$$\| f(x) - \psi_n(x) \|^2 = (h, K_2^{(n+1)} h) \leq \frac{(h,h)}{\lambda_{n+1}^2} \tag{5.24}$$

Since $\lambda_{n+1} \to \infty$, Eq. (5.24) gives

$$\| f(x) - \psi_n(x) \| \to 0 \text{ as } n \to \infty \tag{5.25}$$

Now, we use the relation

$$\| f(x) - \psi(x) \| \leq \| f(x) - \psi_n(x) \| + \| \psi_n(x) - \psi(x) \| \tag{5.26}$$

where $\psi(x)$ is the limit of the series with partial sum ψ_n.

As shown above, the first term on the R.H.S. of (5.26) tends to zero, and to show that the second term of R.H.S. of (5.26) also tends to zero, we proceed as follows:

Since Eq. (5.18) converges uniformly, we have, for an arbitrarily small and positive quantity $\in$, $|\psi_n(x) - \psi(x)| < \in$, when n is sufficiently large.

$\therefore$ $$\| \psi_n(x) - \psi(x) \| < \in (b-a)^{1/2}$$

and hence $$\| \psi_n(x) - \psi(x) \| \to 0$$

Hence, Eq. (5.26) shows that $f(x) = \psi(x)$, and thus, the result follows.

5.8 SCHMIDT'S SOLUTION OF NON-HOMOGENEOUS FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

Consider

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) \, dt \tag{5.27}$$

where $K(x, t)$ is continuous, real and symmetric kernel and λ is not an eigenvalue.

Statement of Hilbert–Schmidt theorem:

Let $F(x)$ be generated from a continuous function $y(x)$ by the operator

$$\lambda \int_a^b K(x,t) \cdot y(t) dt,$$

where $K(x, t)$ is continuous, real and symmetric,

so that $$F(x) = \lambda \int_a^b K(x,t) \cdot y(t) dt \tag{5.28}$$

The function $F(x)$ can be expressed over the interval (a, b) by a linear combination of the normalised eigenfunctions of homogeneous integral equation

$$y(x) = \lambda \int_a^b K(x,t) \cdot y(t) dt \tag{5.29}$$

having $K(x, t)$ as its kernel.

Procedure of solution

From Eq. (5.27), we have

$$y(x) - f(x) = \lambda \int_a^b K(x,t) \cdot y(t) dt \tag{5.30}$$

Since this equation is of the form Eq. (5.28), it follows from Hilbert–Schmidt theorem

$$y(x) - f(x) = \sum_{m=1} a_m \phi_m(x), \quad a \le x \le b \tag{5.31}$$

where $\phi_m(x)$ ($m = 1, 2, 3,...$) are the normalised eigenfunctions of homogeneous integral Eq. (5.29).

Let λ_m be the corresponding eigenvalues of Eq. (5.29), where $\lambda \neq \lambda_m$ for $m = 1, 2, 3,...$

Since $\phi_m(x)$ is normalised, we have

$$\int_a^b \phi_m(x) . \phi_n(x)\, dx = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases} \tag{5.32}$$

Multiplying both sides of Eq. (5.31) by $\phi_m(x)$ and then integrating with respect to x from a to b, we get

$$\int_a^b y(x).\phi_m(x)\, dx - \int_a^b f(x).\phi_m(x)\, dx$$

$$= a_1 \int_a^b \phi_1(x) \phi_m(x) dx + \cdots + a_m \int_a^b \phi_m(x) \cdot \phi_m(x). dx + \cdots. \tag{5.33}$$

Now, let

$$C_m = \int_a^b y(x) \phi_m(x)\, dx \tag{5.34}$$

and

$$f_m = \int_a^b f(x) \phi_m(x)\, dx \tag{5.35}$$

Then, by using Eq. (5.32), Eq. (5.33) provides

$$C_m - f_m = 0 + 0 + \cdots + a_m + 0 \cdots \tag{5.36}$$

And now, multiplying both sides of Eq. (5.27) by $\phi_m(x)$ and then integrating with respect to x from a to b, we get

$$\int_a^b y(x) \phi_m(x)\, dx = \int_a^b f(x) \phi_m(x)\, dx + \lambda \int_a^b \left\{ \int_a^b K(x,t).y(t) dt \right\} \phi_m(x)\, dx$$

Now, using Eqs. (5.34) and (5.35) and interchanging the order of integration

$$C_m = f_m + \lambda \int_a^b y(t) \left\{ \int_a^b K(t,x) \phi_m(x) dx \right\} dt \tag{5.37}$$

Here, we have used the symmetric property of K(x, t).

Further, since $\phi_m(x)$ is eigenfunction corresponding to the eigenvalue λ_m of Eq. (5.29), by definition, we have

$$\phi_m(x) = \lambda_m \int_a^b K(x,t) \cdot \phi_m(t)\, dt$$

[Since eigenfunction has to satisfy the integral equation]

$$\phi_m(x) = \lambda_m \int_a^b K(x,z) \cdot \phi_m(z)\, dz$$

[By changing the variable of integration (t to z)]

$$\phi_m(t) = \lambda_m \int_a^b K(t,z) \cdot \phi_m(z)\, dz$$

[By changing the argument x]

$$\phi_m(t) = \lambda_m \int_a^b K(t,x) \cdot \phi_m(x)\, dx$$

[By again changing the variable of integration]

or $$\int_a^b K(t,x) . \phi_m(x)\, dx = \frac{\phi_m(t)}{\lambda_m} \tag{5.38}$$

Then, Eq. (5.37) gives

$$C_m = f_m + \lambda \int_a^b y(t) \cdot \frac{\phi_m(t)}{\lambda_m} dt = f_m + \lambda \int_a^b \frac{1}{\lambda_m} \cdot y(x) \cdot \phi_m(x)\, dx$$

which upon using Eq. (5.34) shapes as

$$C_m = f_m + \frac{\lambda C_m}{\lambda_m} \tag{5.39}$$

Now, by Eq. (5.36), we have [by eliminating C_m (just by putting C_m)]

$$a_m + f_m = f_m + \frac{\lambda}{\lambda_m}(a_m + f_m)$$

giving $$a_m = \frac{\lambda}{\lambda_m - \lambda} f_m \tag{5.40}$$

Upon substituting this values of a_m in Eq. (5.31), the required solution of Eq. (5.27) is

$$y(x) - f(x) = \sum_{m=1} \left(\frac{\lambda}{\lambda_m - \lambda} f_m \right) \phi_m(x)$$

or $$y(x) = f(x) + \lambda\, \Sigma \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \tag{5.41}$$

Further, using Eq. (5.35) for f_m, we get (variable of integration changed)

$$y(x) = f(x) + \lambda \sum_m \frac{\phi_m(x)}{\lambda_m - \lambda} \int_a^b f(t) \phi_m(t)\, dt$$

or $$y(x) = f(x) + \lambda \int_a^b \left\{ \sum_m \frac{\phi_m(x) \phi_m(t)}{\lambda_m - \lambda} \right\} f(t)\, dt$$

or $$y(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t)\, dt \tag{5.42}$$

where the resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x,t;\lambda) = \sum_m \frac{\phi_m(x).\phi_m(t)}{\lambda_m - \lambda} \tag{5.43}$$

We analyse the following three cases:

Case 1: Unique solution: $\lambda \neq \lambda_m$, a_m is given by Eq. (5.40) to be substituted in Eq. (5.31) to get the unique solution which is given by

$$y(x) = f(x) + \lambda \Sigma \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \tag{5.44}$$

Case 2: No solution: Let the k^{th} eigenvalue be $\lambda_k = \lambda$. Also let $f_k = \int_a^b f(x)\phi_k(x)\,dx \neq 0$, which means that $\phi_k(x)$ is not orthogonal to $f(x)$. Now, because of the presence of the term $\dfrac{f_k\,\phi_m(x)}{\lambda_k - \lambda}$ in the solution, this term is not defined; so, no solution is obtained.

Case 3: Infinitely many solutions: Let $\lambda = \lambda_k$, and $f_k = 0$. Then by Eq. (5.39)

$$C_m = f_m + \frac{\lambda C_m}{\lambda_m}, \quad C_k = 0 + \frac{\lambda}{\lambda} C_k \ \Rightarrow C_k = C_k$$

This being an identity does not impose any restriction on C_k, with the result a_k [refer to Eq. (5.40)] is of form zero/zero and becomes arbitrary. In this situation, we express Eq. (5.41) as follows:

$$y(x) = f(x) + A\phi_k(x) + \lambda \sum_m{}' \frac{\lambda_m}{\lambda_m - \lambda} \phi_m(x) \tag{5.45}$$

where primed Σ implies that we shall omit $m = k$ in the summation and A is an arbitrary constant. The solution given in Eq. (5.45) due to arbitrary nature of A shows that given Eq. (5.27) possesses infinitely many solutions.

EXAMPLE 5.1: By using Hilbert–Schmidt theorem, solve the following symmetric integral equation:

$$y(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)\, y(t)dt$$

Solution: The given equation is $y(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)\, y(t)dt$ (i)
Comparing Eq. (i) with Eq. (5.27), i.e.,

$$y(x) = f(x) + \lambda + \int_a^b K(x,t).y(t)dt \tag{ii}$$

we get $f(x) = (x+1)^2,\ \lambda = 1, K(x,t) = (xt + x^2t^2)$ (iii)

First, we determine the eigenvalues and the corresponding normalised eigenfunctions of Eq. (i) after deleting its non-homogeneous part. So, let

$$y(x) = \lambda \int_{-1}^{1} (xt + x^2t^2) \cdot y(t)dt \tag{iv}$$

$\Rightarrow$
$$y(x) = \lambda x \int_{-1}^{1} t.y(t)\, dt + \lambda x^2 \int_{-1}^{1} t^2 y(t)\, dt \tag{v}$$

Let
$$\int_{-1}^{1} t\, y(t)dt = c_1 \tag{vi}$$

and
$$\int_{-1}^{1} t^2.y(t)\, dt = c_2 \tag{vii}$$

So, Eq. (v) reduces to $y(x) = \lambda c_1.x + \lambda c_2.x^2$ (viii)

and
$$y(t) = \lambda c_1.t + \lambda c_2.t^2 \tag{ix}$$

Then, by Eq. (vi)
$$c_1 = \int_{-1}^{1} t(\lambda c_1.t + \lambda c_2.t^2)\, dt$$

or
$$c_1 = \lambda c_1 \left(\frac{t^3}{3}\right)_{-1}^{1} + \lambda c_2 \left(\frac{t^4}{4}\right)_{-1}^{1}$$

$$c_1 = \frac{2}{3}\lambda c_1 + 0.c_2$$

or
$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right)c_1 + 0.c_2 = 0 \tag{x}$$

Similarly, by Eq. (ix) and (vii)

$$c_2 = \int_{-1}^{1} t^2(\lambda c_1.t + \lambda c_2 t^2)dt$$

or
$$c_2 = c_1\lambda \left(\frac{t^4}{4}\right)_{-1}^{1} + c_2\lambda \left(\frac{t^5}{5}\right)_{-1}^{1}$$

or
$$0.c_1 + \left(1 - \frac{2\lambda}{5}\right) c_2 = 0 \tag{xi}$$

Equation (x) and (xi) will provide non-trivial values of c_1 and c_2 only if

$$D(\lambda) = \begin{vmatrix} 1 - \dfrac{2\lambda}{3} & 0 \\ 0 & 1 - \dfrac{2\lambda}{5} \end{vmatrix} = 0 \tag{xii}$$

Giving λ = 3/2, 5/2 as the required eigenvalues.

Determination of eigenfunction corresponding to λ_1 = 3/2

Putting $\lambda (= \lambda_1) = \dfrac{3}{2}$ in Eq. (x) and (xi), we get

$$0.c_1 + 0.c_2 = 0 \text{ and } 0.c_1 + \left(1 - \frac{2}{5}.\frac{3}{2}\right) c_2 = 0$$

we find that c_2 = 0 and c_1 is arbitrary.

Then, by Eq. (viii), one eigenfunction is $y_1(x) = \frac{3}{2}c_1 x$ or $y_1(x) = x$ by taking $\frac{3c_1}{2} = 1$

Now, the corresponding normalised eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{y_1(x)}{\left[\int_{-1}^{1}\{y_1(x)\}^2 dx\right]^{\frac{1}{2}}} = \frac{x}{\left(\int_{-1}^{1} x^2 dx\right)^{\frac{1}{2}}} = \frac{x\sqrt{6}}{2} \qquad \text{(xiii)}$$

Determination of eigenfunction corresponding to $\boldsymbol{\lambda_2 = \frac{5}{2}}$

Putting $\lambda (= \lambda_2) = \frac{5}{2}$ in Eqs. (x) and (xi), we get

$$0.c_1 + \left(1 - \frac{2}{5}.\frac{5}{2}\right)c_2 = 0 \Rightarrow 0.c_1 + 0.c_2 = 0$$

and

$$\left(1 - \frac{2}{3}.\frac{5}{2}\right)c_1 + 0.c_2 = 0 \Rightarrow \left(1 - \frac{5}{3}\right)c_1 + 0.c_2 = 0$$

We find that here, $c_1 = 0$ and c_2 is arbitrary.

Now, substituting these c_1 and c_2 in Eq. (viii), the eigenfunction corresponding to $\lambda_2 = \frac{5}{2}$ is

$$y_2(x) = 0 + \frac{5}{2}c_2 x^2$$

$$y_2(x) = x^2, \qquad \text{(by taking } \frac{5}{2}c_2 = 1\text{)}$$

Further, the corresponding normalised eigenfunction $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{y_2(x)}{\left[\int_{-1}^{1}\{y_2(x)\}^2 dx\right]^{\frac{1}{2}}} = \frac{x^2}{\left(\int_{-1}^{1} x^4 dx\right)^{\frac{1}{2}}} = \frac{\sqrt{10}}{2}x^2 \qquad \text{(xiv)}$$

Now, by Eqs (iii) and (xiii) [also refer to Eq. (5.35)],

$$f_1 = \int_{-1}^{1} f(x).\,\phi_1(x)dx = \int_{-1}^{1} (x+1)^2\left(\frac{\sqrt{6}x}{2}\right)dx$$

$$f_1 = \frac{\sqrt{6}}{2}\int_{-1}^{1} (x^3 + 2x^2 + x)dx = \frac{\sqrt{6}}{2}\left(\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} = \frac{2\sqrt{6}}{3} \qquad \text{(xiv)}$$

Similarly, by Eqs. (iii) and (xiv),

$$f_2 = \int_{-1}^{1} f(x)\phi_2(x)dx = \int_{-1}^{1}(x+1)^2\frac{\sqrt{10}}{2}x^2 dx$$

$$f_2 = \frac{\sqrt{10}}{2}\left[\frac{x^5}{5} + \frac{2x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} = \frac{8}{15}\sqrt{10} \quad \text{(xv)}$$

Also, from Eq. (iii) $\lambda = 1$ and $\lambda_1 = \frac{3}{2}$, $\lambda_2 = 5/2$;

Thus, we see that $\lambda \neq \lambda_1 \neq \lambda_2$, the integral equation, i.e., Eq. (i) will possess a unique solution, which is given by

$$y(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda}\phi_m(x) \quad \text{(xvi)}$$

$$y(x) = (x+1)^2 + 1\left[\frac{f_1\,\phi_1}{\lambda_1 - \lambda} + \frac{f_2\,\phi_2}{\lambda_2 - \lambda}\right]$$

$$y(x) = (x+1)^2 + \frac{\left(\frac{2}{3}\sqrt{6}\right)\left(\frac{x\sqrt{6}}{2}\right)}{\frac{3}{2} - 1} + \frac{\left(\frac{8\sqrt{10}}{15}\right)\left(\frac{\sqrt{10}}{2}x^2\right)}{\frac{5}{2} - 1}$$

or $\quad y(x) = \frac{25}{9}x^2 + 6x + 1$, is the solution of Eq. (i).

EXAMPLE 5.2: Using Hilbert–Schmidt theorem, find the solution of the following symmetrical integral equation

$$y(x) = (x^2 + 1) + \frac{3}{2}\int_{-1}^{1}(xt + x^2t^2)\cdot y(t)\,dt$$

Solution: The given equations is

$$y(x) = (x^2 + 1) + \frac{3}{2}\int_{-1}^{1}(xt + x^2t^2)\cdot y(t)dt \quad \text{(i)}$$

Comparing Eq. (i) with Eq. (5.27), i.e.,

$$y(x) = f(x) + \lambda\int_a^b K(x,t).\, y(t)\,dt \quad \text{(ii)}$$

We have

$$f(x) = (x^2 + 1),\ \lambda = \frac{3}{2}, K(x,t) = xt + x^2t^2 \quad \text{(iii)}$$

First, we determine the eigenvalues and the corresponding normalised eigenfunctions of Eq. (i) after deleting its non-homogeneous part. Then, it is for

$$y(x) = \lambda\int_{-1}^{1}(xt + x^2t^2)\cdot y(t)dt \quad \text{(iv)}$$

Equation (iv) is same as Eq. (iv) of Example 5.1. So, we take the eigenvalues and the corresponding normalised eigenfunctions and these are

$$\lambda_1 = \frac{3}{2},\ \lambda_2 = \frac{5}{2},\ \phi_1(x) = \frac{x\sqrt{6}}{2},\ \phi_2(x) = \frac{\sqrt{10}x^2}{2}$$

Now, $$f_1(x) = \int_{-1}^{1} f(x)\phi_1(x)dx = \int_{-1}^{1}(x^2+1)\frac{x\sqrt{6}}{2}dx = 0 \qquad \text{(v)}$$

and $$f_2(x) = \int_{-1}^{1} f(x)\phi_2(x)dx$$

$$f_2(x) = \int_{-1}^{1}(x^2+1)\frac{\sqrt{10}x^2}{2}dx = 2\frac{\sqrt{10}}{2}\left[\frac{x^5}{5}+\frac{x^3}{3}\right]_0^1 = \frac{8\sqrt{10}}{15} \qquad \text{(vi)}$$

Here, we find that $\lambda = \frac{3}{2} = \lambda_1$ and $\lambda \neq \lambda_2$, so here, infinitely many solutions will exist [case 3], and then the solution is given by

$$y(x) = f(x) + A\phi_1(x) + \lambda\sum_{m=1}^{2}{}' \frac{f_m}{\lambda_m - \lambda}.\phi_m(x) \qquad \text{(vii)}$$

Here, Σ' means that the term for $m = 1$ must be neglected. So, we get

$$y(x) = f(x) + A\phi_1(x) + \lambda\frac{f_2}{\lambda_2 - \lambda}\phi_2(x)$$

$$y(x) = (x^2+1) + A\left(\frac{x\sqrt{6}}{2}\right) + \frac{3}{2}\,\frac{\frac{(8\sqrt{10})}{15}}{\frac{5}{2}-\frac{3}{2}}\,\frac{x^2\sqrt{10}}{2}$$

or $$y(x) = x^2 + 1 + c_1x + 4x^2,\ c_1 = \frac{A\sqrt{6}}{2}$$

or $y(x) = 5x^2 + c_1x + 1$ is the required solution, c_1 being an arbitrary constant.

EXAMPLE 5.3: Solve the following symmetric integral equation by Hilbert–Schmidt theorem:

$$y(x) = 1 + \lambda\int_0^{\pi}\cos(x+t)\cdot y(t)\,dt$$

Solution: The given equations is

$$y(x) = 1 + \lambda\int_0^{\pi}\cos(x+t)\cdot y(t)dt \qquad \text{(i)}$$

Comparing Eq. (i) with Eq. (5.27), i.e.,

$$y(x) = f(x) + \lambda\int_a^b K(x,t)\,y(t)\,dt \qquad \text{(ii)}$$

We get $$f(x) = 1,\ \lambda = \lambda,\ K(x,t) = \cos(x+t) \qquad \text{(iii)}$$

We first determine the eigenvalues and the corresponding normalised eigenfunctions of Eq. (i) after deleting its non-homogeneous part, i.e., let

$$y(x) = \lambda \int_0^{\pi} \cos(x+t) \cdot y(t)\, dt \tag{iv}$$

or

$$y(x) = \lambda \int_0^{\pi} (\cos x \cos t - \sin x \sin t) \cdot y(t)\, dt$$

or

$$y(x) = \lambda \cos x \int_0^{\pi} \cos t \cdot y(t)\, dt - \lambda \sin x \int_0^{\pi} \sin t \cdot y(t)\, dt \tag{v}$$

Let

$$c_1 = \int_0^{\pi} \cos t \cdot y(t)\, dt \tag{vi}$$

and

$$c_2 = \int_0^{\pi} \sin t \cdot y(t)\, dt \tag{vii}$$

Then, Eq. (v) becomes

$$y(x) = c_1 \lambda \cos x - c_2 \lambda \sin x \tag{viii}$$

and

$$y(t) = c_1 \lambda \cos t - c_2 \lambda \sin t \tag{ix}$$

Then Eq. (vi) gives $c_1 = \int_0^{\pi} \cos t \cdot [c_1 \lambda \cos t - c_2 \lambda \sin t]\, dt$

$$c_1 = \frac{c_1 \lambda}{2} \int_0^{\pi} (1 + \cos 2t)\, dt - \frac{c_2 \lambda}{2} \int_0^{\pi} \sin 2t\, dt$$

$$c_1 = \frac{c_1 \lambda}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{\pi} - \frac{c_2 \lambda}{2} \left[-\frac{\cos 2t}{2} \right]_0^{\pi}$$

$$c_1 = \frac{c_1 \lambda}{2} \cdot \pi$$

$\Rightarrow$

$$c_1 \cdot (2 - \lambda \pi) + 0 \cdot c_2 = 0 \tag{x}$$

Similarly, using Eq. (ix), (vii) gives

$$c_2 = \int_0^{\pi} \sin t \cdot \left[c_1 \lambda \cos t - c_2 \lambda \sin t \right] dt$$

$$c_2 = \frac{c_1 \lambda}{2} \int_0^{\pi} \sin 2t\, dt - \frac{c_2 \lambda}{2} \int_0^{\pi} (1 - \cos 2t)\, dt$$

or

$$c_2 = \frac{c_1 \lambda}{2} \left(\frac{\cos 2t}{2} \right)_0^{\pi} - \frac{\lambda c_2}{2} \left[t - \frac{\sin 2t}{2} \right]_o^{\pi}$$

$$c_2 = 0 - \frac{c_2 \lambda}{2} (\pi - 0)$$

or

$$0 \cdot c_1 + (2 + \lambda \pi) c_2 = 0 \tag{xi}$$

Equations (x) and (xi) will have a non-trivial solution if

$$D(\lambda)=\begin{vmatrix} 2-\lambda\pi & 0 \\ 0 & 2+\lambda\pi \end{vmatrix}=0$$

$$\Rightarrow (2-\lambda\pi)(2+\lambda\pi)=0$$

$$\Rightarrow \lambda=\frac{2}{\pi},\ \frac{-2}{\pi}, \text{(say) } \lambda_1=\frac{2}{\pi}, \lambda_2=\frac{-2}{\pi}$$

So, λ_1 and λ_2 are the eigenvalues.

Determination of eigenfunction corresponding to $\lambda_1=\frac{2}{\pi}$

For this value of $\lambda=\lambda_1=\frac{2}{\pi}$, Eqs. (x) and (xi) give

$$c_1.0 + 0.c_2 = 0 \quad \text{and} \quad 0.c_1 + 4c_2 = 0$$

This means $c_2 = 0$ and c_1 is arbitrary.

Putting these values in Eq. (viii) and recollecting that $\lambda_1=\frac{2}{\pi}$, we have the eigenfunction $y_1(x)$ given by

$$y_1(x)=\frac{2}{\pi}\cdot c_1\cdot\cos x=\cos x\text{, by taking } \frac{2c_1}{\pi}=1$$

The corresponding normalised eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x)=\frac{y_1(x)}{\left[\int_0^\pi \{y_1(x)\}^2 dx\right]^{1/2}}=\frac{\cos x}{\sqrt{\int_0^\pi \cos^2 x\,dx}}=\frac{\cos x}{\sqrt{\int_0^\pi \frac{1+\cos 2x}{2}dx}}$$

$$\phi_1(x)=\frac{\cos x}{\sqrt{\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)_0^\pi}}=\frac{\cos x}{\sqrt{\pi/2}}=\left(\frac{2}{\pi}\right)^{1/2}\cos x \qquad \text{(xii)}$$

Determination of eigenfunction corresponding to $\boldsymbol{\lambda_2=\frac{-2}{\pi}}$

For this value of $\lambda=\lambda_2=\frac{-2}{\pi}$, Eqs. (x) and (xi) give

$$4c_1 + 0.c_1 = 0 \quad \text{and} \quad 0.c_1 + 0.c_2 = 0$$

This means $c_1 = 0$ and c_2 is arbitrary.

Putting these values in Eq. (viii) and recollecting that $\lambda=\frac{-2}{\pi}$, we have the eigenfunction

$$y_2(x)=0-(-2/\pi)\cdot c_2 \sin x$$

or $$y_2(x) = \sin x, \left(\text{Taking } \frac{2c_2}{\pi}=1\right)$$

Then, the corresponding normalised eigenfunction $\phi_2(x)$ is given by

$$\phi_2(x)=\frac{y_2(x)}{\left[\int_0^{\pi}\{y_2(x)\}^2\,dx\right]^{1/2}}=\frac{\sin x}{\sqrt{\int_0^{\pi}\sin^2 x\,dx}}=\sqrt{\frac{2}{\pi}}\sin x \qquad \text{(xiii)}$$

Now, $$f_1(x)=\int_0^{\pi} f(x).\phi_1(x)\,dx=\int_0^{\pi}1.\sqrt{\frac{2}{\pi}}\cos x\,dx=0 \qquad \text{(xiv)}$$

and $$f_2(x)=\int_0^{\pi} f(x)\cdot\phi_2(x)\,dx=\int_0^{\pi}1\cdot\sqrt{\frac{2}{\pi}}\sin x\,dx=2\sqrt{\frac{2}{\pi}} \qquad \text{(xv)}$$

Case 1: Let $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$, then Eq. (i) will possess unique solution given by

$$y(x)=f(x)+\lambda\sum_{m=1}^{2}\frac{f_m}{\lambda_m-\lambda}\phi_m(x)$$

$$y(x)=1+\lambda.\frac{f_1}{\lambda_1-\lambda}\phi_1(x)+\lambda.\frac{f_2}{\lambda_2-\lambda}\phi_2(x)$$

Now, substituting the values of λ_1, f_1, ϕ_1 and λ_2, f_2, ϕ_2 , we get

$$y(x)=1+0+\frac{\lambda}{\left(\frac{-2}{\pi}\right)-\lambda}\cdot 2\left(\frac{2}{\lambda}\right)^{1/2}\cdot\left(\frac{2}{\pi}\right)^{1/2}\sin x \qquad \left(\text{since } f_1=0\right)$$

or $$y(x)=1-\frac{4\lambda\sin x}{2+\pi\lambda} \qquad \text{(xvi)}$$

Case 2: Let $\lambda=\lambda_2=-\dfrac{2}{\pi}$. Since $f_2 \neq 0$, so Eq. (i) possesses no solution.

[Note: f_1 = 0, and denominator of last term becomes zero.]

Case 3: Let λ = λ_1 = $2/\pi$. Since f_1 = 0, there exists infinitely many solutions given by

$$y(x)=f(x)+A\phi_1(x)+\lambda\sum_{m=1}^{2}{}'\,\frac{f_m}{\lambda_m-\lambda}\phi_m(x) \qquad \text{(xvii)}$$

where dash indicates that in the sum, the term corresponding to m = 1 is to be omitted. Thus Eq. (xvii) reduces to

$$y(x)=f(x)+A\,\phi_1(x)+\frac{\lambda f_2}{\lambda_2-\lambda}\phi_2(x)$$

$$y(x)=1+A\left(\frac{2}{\pi}\right)^{1/2}\cos x+\frac{2/\pi}{-\frac{2}{\pi}-\frac{2}{\pi}}\cdot 2\left(\frac{2}{\pi}\right)^{1/2}.\left(\frac{2}{\pi}\right)^{1/2}\sin x$$

$$y(x) = 1 + c\cos x - \frac{2\sin x}{\pi}$$

where, $c = \dfrac{\sqrt{2A}}{\sqrt{\pi}}$ is an arbitrary constant.

EXAMPLE 5.4: Using Hilbert–Schmidt method, solve

$$y(x) = x + \lambda \int_0^1 K(x,t)\cdot y(t)\,dt$$

where,
$$K(x,t) = \begin{cases} x\,(t-1), & 0 \le x \le t \\ t\,(x-1), & t \le x \le 1 \end{cases}$$

Solution: The given equation is

$$y(x) = x + \lambda \int_0^1 K(x,t)\cdot y(t)\,dt \tag{i}$$

where
$$K = \begin{cases} x\,(t-1), & 0 \le x \le t \\ t\,(x-1), & t \le x \le 1 \end{cases} \tag{ii}$$

Comparing Eq. (i) with Eq. (5.27), we get

$$f(x) = x \tag{iii}$$

First, we begin with the determination of eigenvalues and the corresponding eigenfunctions of

$$y(x) = \lambda \int_0^1 K(x,t)\cdot y(t)\,dt \tag{iv}$$

or
$$y(x) = \lambda \left[\int_0^x K(x,t)\cdot y(t)\,dt + \int_x^1 K(x,t)\cdot y(t)\,dt \right]$$

Now, from Eq. (ii),

$$y(x) = \lambda \int_0^x t(x-1)\cdot y(t)\,dt + \lambda \int_x^1 x(t-1)\cdot y(t)\,dt \tag{v}$$

Recollect Leibnitz's rule of differentiation under the sign of integration, which is Eq. (1.20).

[Note the procedure when the kernel is defined in this manner]

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,\xi)\,d\xi = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x}\,d\xi + F[x,H(x)]\frac{dH}{dx} - F[x,G(x)]\frac{dG}{dx}$$

$$y'(x) = \int_0^x \frac{\partial}{\partial x}\{\lambda t\,(x-1).y(t)\}dt + \lambda x\,(x-1).y(x).\frac{d}{dx}(x) - \lambda.0\,(x-1).y(0)\frac{d}{dx}(0)$$

$$+\int_x^1 \frac{\partial}{\partial x}\{\lambda x\,(t-1).y(t)\}dt + \lambda x\,(1-1).y(1).\frac{d}{dx}(1) - \lambda.x\,(x-1).y(x)\frac{d}{dx}(x)$$

or $$y'(x)=\int_0^x \lambda t.y(t)\,dt+\lambda x(x-1).y(x).1-0$$

$$+\int_x^1 \lambda(t-1).y(t)\,dt+0-\lambda x(x-1).y(x).1$$

or $$y'(x)=\int_0^x \lambda t.y(t)\,dt+\int_x^1 \lambda(t-1).y(t)\,dt \tag{vi}$$

Again, differentiating Eq. (vi) with respect to x, we get

$$y''(x)=\int_0^x \frac{\partial}{\partial x}[\lambda t.y(t)]dt+\lambda x.y(x)\frac{d}{dx}(x)-\lambda.0.y(0)\frac{d}{dx}(0)$$

$$+\int_x^1 \frac{\partial}{\partial x}[\lambda(t-1).y(t)]dt+\lambda(1-1).y(1)\frac{d}{dx}(1)-\lambda(x-1)y(x).\frac{d}{dx}(x)$$

or $$y''(x)=0+\lambda x.y(x)-0+0+0-\lambda(x-1).y(x)$$

or $$y''(x)-\lambda.y(x)=0 \tag{vii}$$

by Eq. (v) $$y(0) = 0 \tag{viii}$$

and $$y(1) = 0 \tag{ix}$$

Equation (vii) with Eqs. (viii) and (ix) is Sturm–Liouville problem.*

[refer to $\{r(x).y'\}'+\{q(x)+\lambda p(x)\}y=0 \qquad a\le x\le b$

satisfying $k_1y + k_2y' = 0$ at $x = a$ and $l_1y + l_2y' = 0$ at $x = b$]

We consider the different cases depending upon λ

Case 1: Let $\lambda = 0$, The solution of Eq. (vii) is

$$y(x) = Ax + B \tag{x}$$

Using Eqs. (viii) and (ix), we get

$$B = 0, \quad A + B = 0$$

$\Rightarrow A = 0 = B$, so for $\lambda = 0$, the solution is $y(x) = 0$.

Thus, $\lambda = 0$ is not an eigenvalue and $y(x) = 0$ is not an eigenfunction.

Case 2: Let $\lambda = \mu^2$ ($m \neq 0$). Then, the solution of Eq. (vii) is

$$y = Ae^{\mu x}+Be^{-\mu x};$$

Using Eq. (viii) and (ix), we get

$$A + B = 0, \quad Ae^{\mu}+Be^{-\mu}=0$$

giving $A = 0 = B$.

Thus $y(x) = 0$ is not eigenfunction.

* Refer to Chapter 8 of Differential Equations with applications published by RBD.]

Case 3: Let $\lambda = \mu^2$ ($\mu \neq 0$). Then, the solution of Eq. (vii) is

$$y = A\cos\mu x + B\sin\mu x$$

Using Eq. (viii) and (ix), we get

$$0 = A \text{ and } 0 = A\cos\mu + B\sin\mu$$

giving $B\sin\mu = 0$.

In order that $y(x) = 0$ may not be again a similar trivial solution, we take $B \neq 0$, then $\sin\mu = 0$ giving $\mu = n\pi$, $n \in I^+$

$$\therefore\ \lambda = -\mu^2 = -n^2\pi^2$$

Thus, the required eigenvalues λ_n are

$$\lambda_n = -n^2\pi^2, \quad n = 1, 2, 3, \ldots$$

and

$$y(x) = B\sin n\pi x$$

Taking $B = 1$, we have the eigenfunction

$$y_n(x) = \sin n\pi x, \quad n = 1, 2, 3, \ldots \tag{xi}$$

The normalised eigenfunction $\phi_n(x)$ is given by

$$\phi_n(x) = \frac{y_n(x)}{\left[\int_0^1 \{y_n(x)\}^2\,dx\right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x\,dx\right]^{1/2}}$$

or, $$\phi_n(x) = \sqrt{2}\cdot\sin n\pi x \tag{xii}$$

Now, $$f_n = \int_0^1 f(x).\phi_n(x)\,dx; \quad f(x) = x \qquad \text{[By Eq. (iii)]}$$

$$f_n = \int_0^1 x.\sqrt{2}\sin n\pi x\,dx$$

$$f_n = \sqrt{2}\left[\left\{x.\frac{-\cos n\pi x}{n\pi}\right\}_0^1 - \int_0^1 -\frac{\cos n\pi x}{n\pi}\,dx\right]$$

$$f_n = \sqrt{2}\left\{\frac{-\cos n\pi}{n\pi} + \frac{1}{n\pi}\left(\frac{\sin n\pi x}{n\pi}\right)_0^1\right\}$$

$$f_n = \sqrt{2}\left\{\frac{-(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}.0\right\} = \frac{(-1)^{n+1}\sqrt{2}}{n\pi} \tag{xiii}$$

Now, two cases arise,

Case 1: Let $\lambda \neq \lambda_n$, $n = 1,2,3,\ldots$ i.e., λ is not an eigenvalue. Then Eq. (i) will possess a unique solution, given by

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(x)$$

$$y(x) = x + \lambda \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{2}}{n\pi} \cdot \frac{1}{-n^2\pi^2 - \lambda} \sqrt{2} \sin n\pi x$$

$$y(x) = x + \frac{2\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi x}{n\left(n^2\pi^2 + \lambda\right)} \qquad \text{(xiv)}$$

Case 2: Let $\lambda = \lambda_n = -n^2\pi^2$, $n = 1, 2, 3, \ldots$

Then since from Eq. (xiii), $f_n \neq 0$ for $n = 1, 2, 3, \ldots$

Hence, Eq. (i) will possess no solution.

PRACTICE QUESTIONS WITH INTERMEDIATE RESULTS

1. Solve the symmetric integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x).K(t).y(t)dt$$

Here
$$\lambda_1 = 1 / \int_a^b \{K(x)\}^2 dt, \; y_1(x) = \frac{C.K(x)}{\int_a^b \{K(t)\}^2 dt}$$

$$= K(x) \text{ [By taking } C / \int_a^b \{K(t)\}^2 dt = 1]$$

so
$$\phi_1(x) = \frac{K(x)}{\left[\int_a^b \{K(x)\}^2 dx\right]^{1/2}}$$

$$f_1 = \int_a^b f(x).\, \phi_1(x)dx = \left[\int_a^b f(x).K(x)dx\right] \Big/ \left[\int_a^b \{K(x)\}^2 dx\right]^{\frac{1}{2}}$$

Case 1: Let $\lambda \neq \lambda_1$. Then, we have the unique solution given by

$$y(x) = f(x) + \lambda \sum_{m=1}^{1} \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$y(x) = f(x) + \frac{\lambda}{\lambda_1 - \lambda} f_1 . \phi_1(x)$$

$$y(x) = f(x) + \frac{\lambda}{\left[\int_a^b \{K(x)\}^2 dx\right]^{-1} - \lambda} . \frac{\int_a^b f(x).K(x)dx}{\left[\int_a^b \{K(x)\}^2 dx\right]^{\frac{1}{2}}} . \frac{K(x)}{\left[\int_a^b \{K(x)\}^2 dx\right]^{\frac{1}{2}}}$$

$$y(x) = f(x) + \frac{\lambda.K(x)\int_a^b f(x).K(x)dx}{1 - \lambda\int_a^b \{K(x)\}^2 dx}$$

Case 2: Let $\lambda = \lambda_1$ and assume $f(x)$ is not orthogonal to $\phi_1(x)$, i.e.,

$$f_1 = \int_a^b f(x).\,\phi_1(x)\,dx \neq 0\,,$$

then we have no solution.

Case 3: Let $\lambda = \lambda_1$ and $f_1 = 0$, then, we express the solution as

$$y(x) = f(x) + A\phi_1(x),\ (A \text{ is arbitrary})$$

$$y(x) = f(x) + A\,K(x)\left[\int_a^b \{K(x)\}^2\,dx\right]^{\frac{-1}{2}}$$

$$y(x) = f(x) + c.\ K(x)$$

where, $c = A\left[\int_a^b \{K(x)\}^2\,dx\right]^{\frac{-1}{2}}$ is a constant, though arbitrary.

2. Determine the eigenvalues and the corresponding eigenfunction of the following equation

$$y(x) = x + \lambda\int_0^{2\pi} \sin(x+t).y(t)\,dt$$

Also, find the solution when λ is not an eigenvalue.

Hint: The given equation is

$$y(x) = x + \lambda\int_0^{2\pi} \sin(x+t)\,.y(t)\,dt \tag{i}$$

We first determine the eigenvalues and eigenfunction for

$$y(x) = \lambda\int_0^{2\pi} (\sin x\cos t + \cos x.\sin t).y(t)\,dt$$

$$y(x) = \lambda c_1 \sin x + \lambda c_2 \cos x$$

where, $$c_1 = \int_0^{2\pi} \cos t.y(t)dt \text{ and } c_2 = \int_0^{2\pi} \sin t.y(t)dt$$

Clearly, $$y(t) = \lambda c_1 \sin t + \lambda c_2 \cos t$$

$\therefore$ $$c_1 = \lambda\pi c_2 \Rightarrow c_1 - \lambda\pi c_2 = 0$$

and $$c_2 = \lambda\pi c_1 \Rightarrow \lambda\pi\, c_1 - c_2 = 0$$

$$D(\lambda) = \begin{vmatrix} 1- & \lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{\pi}, \lambda_2 = \frac{-1}{\pi}$$

Determination of eigenfunction for $\lambda_1 = \dfrac{1}{\pi}$

Taking $\lambda = \lambda_1 = \dfrac{1}{\pi}$, we get $c_1 = c_2$ so that

$$y_1(x) = \frac{c_1}{\pi}(\sin x + \cos x) = (\sin x + \cos x)$$

(By taking $\dfrac{c_1}{\pi} = 1$)

$\therefore$ The corresponding normalised eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = y_1(x) \Big/ \left[\int_0^{2\pi} \{y_1(x)\}^2\, dx\right]^{\frac{1}{2}} = (\sin x + \cos x)/\sqrt{2\pi}$$

Determination of eigenfunction for $\boldsymbol{\lambda_2 = -\dfrac{1}{\pi}}$

Taking $\lambda = \lambda_2 = \dfrac{1}{\pi}$, we get $c_1 = -c_2$

so that $y_2(x) = \sin x - \cos x$, (by taking $c_1/\pi = 1$)

Thus, the corresponding normalised eigenfunction $\phi_2(x)$ is given by

$$\phi_2(x) = y_2(x) \Big/ \left[\int_0^{2\pi} \{y_2(x)\}^2 dx\right]^{1/2} = (\sin x - \cos x)/\sqrt{2\pi}$$

Now, $\qquad f_1 = \displaystyle\int_0^{2\pi} f(x) \cdot \phi_1(x) dx = \int_0^{2\pi} \frac{x(\sin x + \cos x)}{\sqrt{2\pi}} dx = -\sqrt{2\pi}$

and $\qquad f_2 = \displaystyle\int_0^{2\pi} f(x) .\, \phi_2(x) dx = \int_0^{2\pi} \frac{x(\sin x - \cos x)}{\sqrt{2\pi}} dx = -\sqrt{2\pi}$

In the question, it is given that $\lambda \neq \lambda_1, \lambda_2$; hence, Eq. (i) will possess unique solution, which is given by

$$y(x) = x + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$y(x) = x + \lambda \frac{f_1 \cdot \phi_1(x)}{\lambda_1 - \lambda} + \lambda \frac{f_2 \cdot \phi_2(x)}{\lambda_2 - \lambda}$$

$$y(x) = x + \frac{\lambda\left(-\sqrt{2\pi}\right)(\sin x + \cos x) / \sqrt{2\pi}}{\dfrac{1}{\pi} - \lambda} + \frac{\lambda\left(-\sqrt{2\pi}\right)(\sin x - \cos x)/\sqrt{2\pi}}{-\dfrac{1}{\pi} - \lambda}$$

or $y(x) = x - \dfrac{2\lambda^2\pi^2 \sin x}{1 - \lambda^2\pi^2} - \dfrac{2\lambda\pi \cos x}{1 - \lambda^2\pi^2}$

3. Solve the symmetric integral equation

$$y(x) = e^x + \lambda \int_0^1 K(x,t) . y(t) dt$$

where, $$K(x,t) = \begin{cases} \dfrac{\sinh x . \sinh (t-1)}{\sinh 1}, & 0 \le x \le t \\ \dfrac{\sinh t . \sinh (x-1)}{\sinh 1}, & t \le x \le 1 \end{cases}$$

Hint: The given equation is

$$y(x) = e^x + \lambda \int_0^1 K(x,t) \cdot y(t) dt \quad \text{(i)}$$

where, $$K(x,t) = \begin{cases} \dfrac{\sinh x . \sinh (t-1)}{\sinh 1}, & 0 \le x \le t \quad \text{[ii(a)]} \\ \dfrac{\sinh t . \sinh (x-1)}{\sinh 1}, & t \le x \le 1 \quad \text{[ii(b)]} \end{cases}$$

Here, $$f(x) = e^x$$

First, we determine the eigenvalues and eigenfunction for the homogeneous integral equation

$$y(x) = \lambda \int_0^1 K(x,t) . y(t) dt \quad \text{(iii)}$$

or $$y(x) = \lambda \int_{t=0}^{x} \frac{\sinh t \sinh(x-1)}{\sinh 1} . y(t) dt + \lambda \int_{t=x}^{1} \frac{\sinh x \sinh(t-1)}{\sinh 1} . y(t) dt \quad \text{(iv)}$$

Following Leibnitz's rule, differentiating Eq. (iv), we get

$$y'(x) = \lambda \int_0^x \frac{\sinh t \cosh (x-1)}{\sinh 1} . y(t) dt + \frac{\lambda \sinh x \sinh (x-1) y(x)}{\sinh 1} . 1$$

$$+ \lambda \int_x^1 \frac{\cosh x \sinh(t-1)}{\sinh 1} . y(t) dt - \frac{\lambda \sinh x . \sinh (x-1)}{\sinh 1} y(x) . 1$$

Again differentiating,

$$y''(x) = \lambda \int_0^x \frac{\sinh t \sinh (x-1)}{\sinh 1} y(t) dt + \frac{\lambda \sinh x . \cosh (x-1)}{\sinh 1} y(x) . 1$$

$$+ \lambda \int_x^1 \frac{\sinh x - \sinh (t-1)}{\sinh 1} y(t) dt - \frac{\lambda \cosh x . \sinh (x-1)}{\sinh 1} y(x) . 1$$

Using Eq. (iv), it reduces to

$$y''(x) = y(x) + \frac{\lambda y(x)}{\sinh 1}\left[\sinh x \cosh\,(x-1) - \cosh x.\sinh\,(x-1)\right]$$

or $$y''(x) = y(x) + \lambda.y(x) \Rightarrow y'' - (1+\lambda)y = 0 \tag{v}$$

with $y(0) = 0$ [obtained by putting $x = 0$ in Eq. (iv)] (vi)

and $y(1) = 0$ [obtained by putting $x = 1$ in Eq. (iv)] (vii)

For the solution of Eq. (v), we consider the following three cases:

Case 1: Let $1 + \lambda = 0 \Rightarrow \lambda = -1$, then by Eq. (v) $y'' = 0$, giving $y = Ax + B$. Now, by Eq. (vi), $B = 0$ and by Eq. (vii), $A + B = 0$, which means both $A = 0$, $B = 0$,

Therefore, $y = 0$.

Thus it is not an eigenfunction, and correspondingly, $\lambda = -1$ is not an eigenvalue.

Case 2: Let $\lambda + 1 = \mu^2 (\mu \neq 0)$. Then, the general solution of Eq. (v) is

$$y(x) = Ae^{\mu x} + Be^{-\mu x}.$$

For A and B, we use Eq. (vi) and Eq. (vii) giving

$$0 = A + B \text{ and } 0 = Ae^{\mu} + Be^{-\mu};$$

and these equations provide $A = 0 = B$.

Thus, $y(x) = 0$ is not eigenfunction and $\lambda = \mu^2 - 1$ is not an eigenvalue.

Case 3: Let $\lambda + 1 = -\mu^2 (\mu \neq 0)$. Then Eq. (v) has its general solution as

$$y(x) = A \cos \mu x + B \sin \mu x$$

Using Eqs. (vi) and (vii), we get

$$A = 0,\ B \sin \mu = 0$$

Clearly, $B \neq 0$ (otherwise we reach at the previous cases)

$\therefore \quad \mu = n\pi\ ,\ n \in I$ (an integer)

$\therefore \quad 1 + \lambda = -\mu^2 = -n^2\pi^2$ or $\lambda = -\left(1 + n^2\pi^2\right)$

and so, the required eigenvalues are $\lambda_n = -\left(1 + n^2\pi^2\right), n = 1, 2, 3, \ldots.$

Now, taking $B = 1$, the eigenfunctions are

$$y_n(x) = \sin n\pi x, \quad n = 1,2,3, \ldots$$

The normalised eigenfunction $\phi_n(x)$ is given by

$$\phi_n(x) = \frac{y_n(x)}{\left[\int_0^1 \{y_n(x)\}^2\, dx\right]^{\frac{1}{2}}} = \frac{\sin n\pi x}{\sqrt{\int_0^1 \sin^2 n\pi x dx}} = \sqrt{2}\sin n\pi x \tag{viii}$$

Then, $$f_n = \int_0^1 f(x)\,.\,\phi_n(x)dx = \int_0^1 e^x \phi_n(x)\,dx$$

$$f_n = \int_0^1 e^x\,.\left(\sqrt{2}\sin n\pi x\right)dx$$

$$f_n = \frac{\sqrt{2}}{1+n^2\pi^2}\left[e^x(\sin n\pi x - n\pi\cos n\pi x)\right]_0^1$$

$$f_n = \frac{\sqrt{2}}{1+n^2\pi^2}[-e n\pi\ \cos n\pi - (-n\pi)]$$

or $$f_n = \frac{n\pi\sqrt{2}}{1+n^2\pi^2}\left[1-e(-1)^n\right],\ n = 1,2,3 \qquad \text{(ix)}$$

Here, we consider two situations:

(a) Let $\lambda \neq \lambda_n$ (means λ is not an eigenvalue). Then Eq. (i) will possess a unique solution given by

$$y(x) = f(x) + \lambda\sum_{n=1}^{\infty}\frac{f_n}{\lambda_n - \lambda}\,\phi_n(x)$$

$$y(x) = e^x + \lambda\sum_{n=1}^{\infty}\frac{n\pi\sqrt{2}}{1+n^2\pi^2}[1-e(-1)^n]\frac{\sqrt{2}\,\sin n\pi x}{-(1+n^2\pi^2)-\lambda}$$

or, $$y(x) = e^x - 2\pi\lambda\sum_{n=1}^{\infty}\frac{n[1-e(-1)^n]\sin n\pi x}{[1+n^2\pi^2][1+\lambda+n^2\pi^2]}$$

(b) Let $$\lambda = \lambda_n = -1-n^2\pi^2,\ n = 1,2,3,\ldots.$$

Then, from Eq. (ix), $f_n \neq 0$ for $n = 1, 2, 3, \ldots$

Hence, Eq. (i) will possess no solution.

4. Using Hilbert–Schmidt theorem, solve the following symmetric integral equations:

(a) $y(x) = x + \int_0^1 (x+1)\,.y(t)dt, \lambda \neq \lambda_1, \lambda_2$

$$\left[\textbf{Ans:}\ y(x) = \frac{(6\lambda-12)x-4\lambda}{\lambda^2+12\lambda-12}\right]$$

(b) $y(x) = \left(1-x\sqrt{3}\right) + \left(-6+4\sqrt{3}\right)\int_0^1 (x+t)y(t)dt$

[**Ans:** $y(x) = \left(1-x\sqrt{3}\right) + C\left(1+\sqrt{3}\right) - \left(1+\dfrac{3x}{2}\right)$, C being an arbitrary constant.]

EXERCISE 5.1

Using Hilbert–Schmidt theorem, solve the following symmetric integral equations:

1. $y(x) = \frac{1}{2} - x + \int_0^1 y(t)\, dt,$

2. $y(x) = x + \lambda \int_0^1 xs\, y(s) ds,$

3. $y(x) = x + \lambda \int_0^1 y(s) ds,\ \lambda \neq 1$

4. $y(x) = (1 + x\sqrt{3}) - (6 + 4\sqrt{2}) \int_0^1 (x + t) \cdot y(t) dt$

5. $y(x) = \cos \pi x + \lambda \int K(x, \xi) \cdot y(\xi) d\xi$

$$\text{where, } k(x,\xi) = \begin{cases} \xi(x+1) & 0 \le x \le \xi \\ x(\xi+1) & \xi \le x \le 1 \end{cases}$$

Answers:

1. $[y(x) = \frac{1}{2} - x + c]$

3. $[y(x) = x + \lambda/\{2(1 - \lambda)]$

4. $[\, y(x) = (1 + x\sqrt{3}) - (1 + 3x/2) + c(1 - x\sqrt{3})\,]$

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CHAPTER 6

Solution of Integral Equations of the Second Kind by Successive Approximation

6.1 INTRODUCTION

In the previous chapters of the book, the solution of the integral equations was mainly focused upon Fredholm integral equation. In this chapter the development is for Volterra integral equation also. Moreover, where the solution is not possible in closed form, successive approximate method is also discussed. Apart from the various theorems, the chapter includes three parts—(a) iterated kernels, (b) resolvent kernel, and (c) solution of integral equations by applying resolvent kernel.

6.2 ITERATED KERNEL OR FUNCTION

1. Let us consider the following Fredholm integral equation of the second kind:

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot y(t)dt \tag{6.1}$$

Then, the iterated kernels $K_n(x,t), n = 1, 2, 3, ...,$ are defined as follows:

$$K_1(x,t) = K(x,t) \qquad [6.2(a)]$$

$$\left.\begin{aligned} &\text{and} \quad K_n(x,t) = \int_a^b K(x,z) \cdot K_{n-1}(z,t)dz, \quad n = 2,3,... \\ &\text{or} \quad K_n(x,t) = \int_a^b K_{n-1}(x,z) \cdot K(z,t)dz, \quad n = 2,3,... \end{aligned}\right\} \qquad [6.2(b)]$$

2. Let the Volterra integral equation of the second kind be

$$y(x) = f(x) + \lambda \int_a^x K(x,t) \cdot y(t)dt \tag{6.3}$$

Then, the iterated kernels $K_n(x,t), n = 1, 2, 3, ...,$ are defined as follows:

$$K_1(x,t) = K(x,t) \qquad [6.4(a)]$$

and
$$\left.\begin{aligned} K_n(x,t) &= \int_t^x K(x,z)\cdot K_{n-1}(z,t)dz, \qquad n = 2,3,\ldots \\ \text{or} \quad K_n(x,t) &= \int_t^x K_{n-1}(x,z)\cdot K(z,t)dz, \qquad n = 2,3,\ldots \end{aligned}\right\} \qquad [6.4(b)]$$

6.3 RESOLVENT KERNEL OR RECIPROCAL KERNEL

We consider the solution of Fredholm integral equation of the second kind.

$$y(x) = f(x) + \lambda\int_a^b K(x,t)\cdot y(t)dt \qquad (6.5)$$

and let it take the following form

$$y(x) = f(x) + \lambda\int_a^b R(x,t;\lambda)\cdot f(t)dt \qquad [6.6(a)]$$

or
$$y(x) = f(x) + \lambda\int_a^b \Gamma(x,t;\lambda)\cdot f(t)dt \qquad [6.6(b)]$$

Here, $R(x, t; \lambda)$ or $\Gamma(x, t; \lambda)$ is known as resolvent kernel of Eq. (6.5). Analogously, we have the resolvent kernel for Volterra integral Eq. (6.3). We consider the following theorem without proof:

Theorem: The m^{th} iterated kernel $K_m(x, t)$ satisfies the relation

$$K_m(x,t) = \int_a^b K_r(x,y)\cdot K_{m-r}(y,t)\,dy$$

where, r is any positive integer less than m.

6.4 SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE SUBSTITUTION

Theorem: Let $y(x) = f(x) + \lambda\int_a^b K(x,t)\cdot y(t)dt$ (6.7)

be the given Fredholm integral equation of the second kind. Let

1. $K(x, t) \not\equiv 0$ be real and continuous in the rectangle R, for which $a \le x \le b$, $a \le t \le b$. Also, assume $|K(x, t)| \le M$ in R.
2. $f(x) \ne 0$ be real and continuous in the interval I for which $a \le x \le b$. Also, assume $|f(x)| \le N$ in I.
3. λ be a constant such that $|\lambda| < 1/M(b - a)$. Then (6.5) has a unique solution in I and this solution is given by the absolutely and uniformly convergent series*

$$y(x) = f(x) + \lambda\int_a^b K(x,t)f(t)dt + \lambda^2\int_a^b K(x,t)\cdot\int_a^b K(t,t_1)\cdot f(t)dt_1\,dt + \ldots \qquad (6.8)$$

---

* A convergent infinite series has a sum. But, if we differentiate (or integrate) its all the terms, then this sum may not be equal to the derivative (or integral) of sum. However, if the series is absolutely and uniformly convergent, then it is possible. To check this characteristics, we check if the modulus of the nth term is less than a definite quantity.

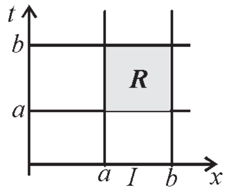

Figure. 6.1 Region R for $a \le n \le b$ and $a \le t \le b$.

Proof: Rewriting Eq. (6.7) as

$$y(x) = f(x) + \lambda \int_a^b K(x, t_1) \cdot y(t_1) dt_1 \tag{6.9}$$

Now, replacing x by t in Eq. (6.9), we have

$$y(t) = f(t) + \lambda \int_a^b K(t, t_1) \cdot y(t_1) dt_1 \tag{6.10}$$

Substituting this $y(t)$ in R.H.S. of Eq. (6.7), we get

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot \left\{ f(t) + \lambda \int_a^b K(t,t_1) \cdot y(t_1) dt_1 \right\} dt$$

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) \cdot y(t_1)\, dt_1 dt \tag{6.11}$$

Now, replacing t_1 by t_2 and t by t_1 in Eq. (6.10), we have

$$y(t_1) = f(t_1) + \lambda \int_a^b K(t_1, t_2) \cdot y(t_2)\, dt_2 \tag{6.12}$$

Substituting this value of $y(t_1)$ in R.H.S. of Eq. (6.11), we get

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot f(t)\, dt +$$

$$\lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) \left\{ f(t_1) + \lambda \int_a^b K(t_1,t_2) \cdot y(t_2)\, dt_2 \right\} dt_1 dt$$

or $$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) \cdot f(t_1)\, dt_1 dt$$

$$+\lambda^3 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2) \cdot y(t_2) \cdot dt_2 \cdot dt_1 \cdot dt \tag{6.13}$$

Proceeding likewise, we have

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1) dt_1 dt$$

$$+ \cdots + \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \cdots \int_a^b K(t_{n-2}, t_{n-1}) \cdot f(t_{n-1})\, dt_{n-1} \cdots dt_1 dt$$

$$+R_{n+1}(x) \tag{6.14}$$

where, $$R_{n+1}(x) = \lambda^{n+1} \int_a^b K(x,t) \int_a^b K(t,t_1) \cdots \int_a^b K(t_{n-1},t_n) \cdot y(t_n) dt_n \cdots dt_1 dt \tag{6.15}$$

We now consider the following infinite series:

$$f(x)+\lambda\int_a^b K(x,t)f(t)dt+\lambda^2\int_a^b K(x,t)\int_a^b K(t,t_1)f(t_1)\cdot dt_1 dt+\cdots \tag{6.16}$$

Following the assumptions (1) and (2), each term of Eq. (6.16) is continuous in *I*; and it is thus obvious that Eq. (6.16) is also continuous, provided it converges uniformly in *I*.

Let $U_n(x)$ represent the general term of Eq. (6.16), which we express as

$$U_n(x)=\lambda^n\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-2},t_{n-1})\cdot f(t_{n-1}).dt_{n-1}\cdots dt_1 dt \tag{6.17}$$

Taking its modulus,

$$|U_n(x)|=|\lambda^n\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}\cdots dt_1 dt|$$

Therefore,

$$|U_n(x)|\le|\lambda|^n N\cdot M^n(b-a)^n \quad \text{[By the conditions of theorem]} \tag{6.18}$$

And Eq. (6.15) converges only when

$$|\lambda|\cdot M\cdot(b-a)\le 1$$

or

$$|\lambda|<1/M(b-a) \tag{6.19}$$

and it holds good due of condition 3.

It thus means that if Eq. (6.7) has a continuous solution, it is given by Eq. (6.14). If $y(x)$ is continuous in *I*, $|y(x)|$ must have a maximum value, say *Y*. Thus,

$$|y(x)|\le Y \tag{6.20}$$

Now, from Eq. (6.15),

$$|R_{n+1}(x)|=|\lambda^{n+1}\int_a^b K(x,t)\int_a^b K(t,t_1)\cdots\int_a^b K(t_{n-1},t_n)y(t_n)dt_n\cdots dt_1 dt|$$

or

$$|R_{n+1}(x)|\le|\lambda|^{n+1}\cdot Y\cdot M^{n+1}(b-a)^{n+1}$$

Using condition 1 and Eq. (6.20) and now due to condition 3

$$\lim_{n\to\infty} R_{n+1}(x)=0$$

It thus means that function $y(x)$ satisfying Eq. (6.14) is the continuous function given by Eq. (6.16) or Eq. (6.8).

6.5 SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE SUBSTITUTIONS

Theorem: Let $y(x)=f(x)+\lambda\int_a^x K(x,t)\cdot y(t)dt$ (6.21)

be the Volterra integral equation of the second kind. Let

1. Kernal $K(x, t) \not\equiv 0$ be real and continuous in rectangle R for which $a \le x \le b$, $a \le t \le b$. Also, assume
$$|K(x, t)| \le M \text{ in } R \tag{6.22}$$
2. $f(x) \not\equiv 0$, be real and continuous in the interval I, for which
$$a \le x \le b. \text{ Also, assume } |f(x)| \le N \text{ in } I \tag{6.23}$$
3. λ be constant. Then Eq. (6.21) has a unique continuous solution is I and this solution is given by absolutely and uniformly convergent series

$$y(x) = f(x) + \lambda\int_a^x K(x,t)f(t)dt + \lambda^2\int_a^x K(x,t)\cdot\int_a^t K(t_1,t_2)\cdot f(t_1)dt_1dt + \cdots \tag{6.24}$$

Proof: Equation (6.21) can be expressed as (let $t = t_1$)

$$y(x) = f(x) + \lambda\int_a^x K(x,t_1)\cdot y(t_1)\,dt_1 \tag{6.25}$$

Now, replacing x by t in Eq. (6.25), we get

$$y(t) = f(t) + \lambda\int_a^t K(t,t_1)\cdot y(t_1)\,dt_1 \tag{6.26}$$

Substituting this value of $y(t)$ in R.H.S. of Eq. (6.21), we obtain

$$y(x) = f(x) + \lambda\int_a^x K(x,t)\left\{f(t) + \lambda\int_a^t K(t,t_1)\cdot y(t_1)\,dt_1\right\}dt$$

or $$y(x) = f(x) + \lambda\int_a^x K(x,t)\cdot f(t)dt + \lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\cdot y(t_1)dt_1\,dt \tag{6.27}$$

Now, we find $y(t_1)$ to put in Eq. (6.27), and for this, we replace t_1 by t_2, and then, t by t_1 in Eq. (6.26), we get

$$y(t_1) = f(t_1) + \lambda\int_a^{t_1} K(t_1,t_2)\cdot y(t_2)dt_2$$

Substituting this $y(t_1)$ in Eq. (6.27), we have

$$y(x) = f(x) + \lambda\int_a^x K(x,t)f(t)dt$$
$$+\lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\left[f(t_1) + \lambda\int_a^{t_1} K(t_1,t_2)\cdot y(t_2)dt_2\right]dt_1\cdot dt$$

or $$y(x) = f(x) + \lambda\int_a^x K(x,t)\cdot f(t)dt + \lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\cdot f(t_1)\,dt_1\cdot dt$$
$$+\lambda^3\int_a^x K(x,t)\int_a^t K(t,t_1)\int_a^{t_1} K(t_1,t_2)\cdot y(t_2)dt_2dt_1dt \tag{6.28}$$

Proceeding similarly, we find

$$y(x) = f(x) + \lambda\int_a^x K(x,t)\cdot f(t)dt + \lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\cdot f(t_1)\,dt_1\cdot dt$$
$$+\cdots + \lambda^n\int_a^x K(x,t)\int_a^t K(t,t_1)\cdots\int_a^{t_{n-2}} K(t_{n-2},t_{n-1})\cdot f(t_{n-1})\,dt_{n-1}\cdots dt_1dt + R_{n+1}(x) \tag{6.29}$$

where,

$$R_{n+1}(x) = \lambda^{n+1}\int_a^x K(x,t)\int_a^t K(t,t_1)\cdots\int_a^{t_{n-1}} K(t_{n-1},t_n)\cdot y(t_n)\,dt_n\cdots dt_1 dt \quad (6.30)$$

We now consider the following infinite series:

$$y(x) + \lambda\int_a^x K(x,t)\cdot f(t)dt + \lambda^2\int_a^x K(x,t)\cdot\int_a^t K(t,t_1)f(t_1)\,dt_1 + \cdots \quad (6.31)$$

Now, due to conditions (1) and (2), each term of Eq. (6.31) is continuous in *I*. It makes Eq. (6.31) also continuous in *I*, provided it converges uniformly in *I*. Let $V_n(x)$ be the general term of Eq. (6.31), given by

$$V_n(x) = \lambda^n\int_a^x K(x,t)\int_a^t K(t,t_1)\cdots\int_a^{t_{n-2}} K(t_{n-2},t_{n-1}).f(t_{n-1})\,dt_{n-1}\cdots dt_1 dt \quad (6.32)$$

Then, $|V_n(x)| \le |\lambda|^n\cdot N\cdot M^n\dfrac{(x-a)^n}{n!}$

[Here, we have applied conditions 1 and 2 over mod of Eq. (6.32)]

or $$|V_n(x)| \le |\lambda|^n\cdot N\cdot M^n\frac{(b-a)^n}{n!}, \qquad a\le x\le b$$

or $$|V_n(x)| \le |\lambda|^n\cdot N\cdot[M(b-a)]^n/n! \quad (6.33)$$

Now, from Eq. (6.32), it is clear that Eq. (6.31) is convergent for all λ, N, M, $(b-a)$; thus, from Eq. (6.33), it is followed that Eq. (6.31) is convergent obsolutely and uniformly.

So, if Eq. (6.21) has a continuous solution, it must be expressed by Eq. (6.29). If $y(x)$ is continuous in *I*, $|y(x)|$ must have a maximum value, say Y.

or $$|y(x)| \le Y \quad (6.34)$$

Now, from Eq. (6.30), $|R_{n+1}(x)| \le |\lambda|^{n+1}\,Y\cdot M^{n+1}\dfrac{(x-a)^{n+1}}{(n+1)!}$

$$|R_{n+1}(x)| \le |\lambda|^{n+1}\,Y\cdot M^{n+1}\frac{(b-a)^{n+1}}{(n+1)!} \qquad \because a\le x\le b$$

Hence, $\lim\limits_{n\to\infty} R_{n+1}(x) = 0$

It thus follows that function $y(x)$ satisfying Eq. (6.29) is the continuous function given by Eq. (6.24) or Eq. (6.31). It thus proves the theorem.

6.6 SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE APPROXIMATIONS: ITERATIVE METHOD (ITERATIVE SCHEME) NEUMANN SERIES

We consider the following Fredholm integral equation of the second kind:

$$y(x) = f(x) + \lambda\int_a^b K(x,t)\cdot y(t)dt \quad (6.35)$$

For the required solution $y(x)$, we begin with zero order approximation. So, let

$$y_0(x) = f(x) \tag{6.36}$$

Now, if $y_n(x)$ and $y_{n-1}(x)$ represent the n^{th} and $(n-1)^{th}$ order approximations respectively, then these are connected by

$$y_n(x) = f(x) + \lambda \int_a^b K(x,t) \cdot y_{n-1}(t)\, dt \tag{6.37}$$

Further, the iterated kernels are given by

$$K_1(x,t) = K(x,t) \qquad [6.38(a)]$$

and $$K_n(x,t) = \int_a^b K(x,z) \cdot K_{n-1}(z,t) dz, \qquad n = 2,3,4... \qquad ....[6.38(b)]$$

Let $n = 1$ in Eq. (6.37). The first order appoximation is given by

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) y_0(t) dt \tag{6.39}$$

Now, since $y_0(t) = f(t)$ by [using Eq. (6.36), we have [by Eq. (6.39)]

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt \tag{6.40}$$

Let $n = 2$ in Eq. (6.37). The second order approximation $y_2(x)$ is

$$y_2(x) = f(x) + \lambda \int_a^b K(x,t) \cdot y_1(t) dt$$

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) \cdot y_1(z) dz \tag{6.41}$$

Now, substituting $y_1(z)$ from Eq. (6.40) by replacing x by z, we get

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) \left[f(z) + \lambda \int_a^b K(z,t).f(t) dt \right] dz$$

or $$y_2(x) = f(x) + \lambda \int_a^b K(x,z) \cdot f(z) dz + \lambda^2 \int_a^b K(x,z) \int_a^b K(z,t) \cdot f(t) dt\, dz \tag{6.42}$$

(also replacing z by t in the second term)

Now, interchanging the order of integration (in the last term), we have

$$y_2(x) = f(x) + \lambda \int_a^b K(x,t) \cdot f(t)\, dt + \lambda^2 \int_a^b f(t) \cdot \left[\int_a^b K(x,z) \cdot K(z,t) dz \right] dt$$

Now, using Eqs. [6.38(a)] and [(6.38(b)] in the second and third term, respectively, we obtain

$$y_2(x) = f(x) + \lambda \int_a^b K_1(x,t) \cdot f(t) dt + \lambda^2 \int_a^b K_2(x,t) \cdot f(t) dt$$

or $$y_2(x) = f(x) + \sum_{m=1}^{2} \lambda^m \int_a^b K_m(x,t) \cdot f(t) dt \tag{6.43}$$

Proceeding similarly, we generalise it and write

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,t) \cdot f(t)dt \tag{6.44}$$

Now, upon taking the limit $n \to \infty$, we find the Neumann series

$$y(x) = \lim_{n\to\infty} y_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,t) \cdot f(t)dt \tag{6.45}$$

Resolvent Kernal (or Reciprocal Kernel)

To determine $R(x, t; \lambda)$ in terms of iterated kernel $K_m(x, t)$, we change the order of integration and summation in Eq. (6.45), and obtain

$$y(x) = f(x) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \right] f(t)dt \tag{6.46}$$

Comparing Eq. (6.46) with

$$y(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) \cdot f(t)dt \tag{6.47}$$

we have

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \tag{6.48}$$

6.7 RESOLVENT KERNEL OF A FREDHOLM INTEGRAL EQUATION

Theorem: Let $R(x, t; \lambda)$ be the resolvent kernel of the following Fredholm integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t)dt \tag{6.49}$$

Then

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,z) \cdot R(z,t;\lambda)dz \tag{6.50}$$

Proof: We know (by Eq. (6.48) that

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

where the iterated kernels are

$$K_1(x,t) = K(x,t) \qquad [6.51(a)]$$

and

$$K_m(x,t) = \int_a^b K(x,z) K_{m-1}(z,t)dz, \qquad m = 2, 3, \ldots \qquad [6.51(b)]$$

Then, Eq (6.48) can be expressed as

$$R(x,t;\lambda) = K_1(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} K_m(x,t)$$

Putting values in R.H.S. using Eq. [6.51(a)] for $K_1(x, t)$ and [6.51(b)] for $K_m(x, t)$, we get

$$R(x,t;\lambda) = K(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^b K(x,z) \cdot K_{m-1}(z,t)\, dz$$

Setting n for $m - 1$,

$$R(x,t;\lambda) = K(x,t) + \sum_{n=1}^{\infty} \lambda^{n} \int_a^b K(x,z) \cdot K_{n}(z,t)\, dz$$

$$R(x,t;\lambda) = K(x,t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K(x,z) \cdot K_{m}(z,t)\, dz$$

[Replacing n by m]

Now, interchanging the signs of summation and integration

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(z,t) \right] K(x,z)\, dz$$

Using Eq. (6.48) in the integrnal part,

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b [R(z,t;\lambda)] \cdot K(x,z)\, dz$$

or $$K(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,z) \cdot R(z,t;\lambda)\, dz$$

6.8 ILLUSTRATIONS BASED ON THE SOLUTION OF FREDHOLM INTEGRAL EQUATION BY SUCCESSIVE APPROXIMATIONS (ITERATIVE METHOD)

The procedure shall be in three stages. First we shall learn to find iterated kernel, i.e., $K_m(x, t)$ from the given kernel $K(x, t)$, refer Eq. (6.51). In the second stage, we shall find resolvent kernel $R(x, t; \lambda)$, and in the third stage, we shall solve Eq. (6.52) through $R(x, t; \lambda)$.

Type 1. ***Determination of iterated kernel***

We will find the iterated kernel for

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot y(t)\, dt \tag{6.52}$$

EXAMPLE 6.1: Find the iterated kernel for following kernels:

(a) $K(x,t) = \sin(x - 2t), 0 \le x \le 2\pi, 0 \le t \le 2\pi$

(b) $K(x,t) = e^x \cos t,\ 0 \le x \le \pi,\ 0 \le t \le \pi$

(c) $K(x,t) = x + \sin t,\ -\pi \le x \le \pi,\ -\pi \le t \le \pi$

(d) $K(x,t) = x - t,\ 0 \le x \le 1,\ 0 \le t \le 1$

Solution: (a) Iterated kernel $K_n(x,t)$ may be represented as [using Eq. (6.2)]

$$K_1(x,t) = K(x,t) \quad \text{(i)}$$

and
$$K_n(x,t) = \int_a^b K(x,z)\cdot K_{n-1}\, z,t)\,dz, \qquad n = 2,3,... \quad \text{(ii)}$$

Here,
$$a = 0, b = 2\pi.$$

So,
$$K_1(x,t) = K(x,t) = \sin(x-2t) \quad \text{(iii)}$$

Let $n = 2$ in Eq. (ii), we have

$$K_2(x,t) = \int_0^{2\pi} K(x,z)\cdot K_1(z,t)\,dz$$

Using Eq. (iii)

$$K_2(x,t) = \int_0^{2\pi} \sin(x-2z)\cdot \sin(z-2t)\,dz$$

$$K_2(x,t) = \frac{1}{2}\int_0^{2\pi} [\cos(x+2t-3z) - \cos(x-2t-z)]\,dz$$

$$K_2(x,t) = \frac{1}{2}\left[-\frac{1}{3}\sin(x+2t-3z) + \sin(x-2t-z)\right]_0^{2\pi}$$

So,
$$K_2(x,t) = 0 \quad \text{(iv)}$$

Let $n = 3$ in Eq. (ii). Then, we have

$$K_3(x,t) = \int_0^{2\pi} K(x,z)\cdot K_2(z,t)dz = 0 \qquad \text{[due to Eq. (iv)]}$$

Thus, $K_1(x,t) = \sin(x - 2t)$

and $K_n(x,t) = 0$, $n = 2,3, ...$

(b) Here, with $K(x, t) = e^x \cos t$ $a = 0$, $b = \pi$ using Eq. (i),

$$K_1(x,t) = K(x,t) = e^x \cos t \quad \text{(v)}$$

and using Eq. (ii) for $n = 2$,

$$K_2(x,t) = \int_0^{\pi} K(x,z)\cdot K_1(z,t)dz$$

for $K_1(z,t)$, we use Eq. (v).

$$K_2(x,t) = \int_0^{\pi} e^x \cos z\cdot e^z \cos t\,dz$$

$$= e^x \cos t.\int_0^{\pi} e^x \cdot \cos z\ dz$$

$$= e^x \cos t\left\{\frac{e^z}{1^2+1^2}[\cos z + \sin z]\right\}_0^{\pi}$$

or
$$K_2(x,t) = (-1)^1\,\frac{1+e^{\pi}}{2}(e^x\cdot \cos t) \quad \text{(vi)}$$

Now, for $K_3(x,t)$, again using Eq. (ii) for $n = 3$, we get

$$K_3(x,t) = \int_0^{\pi} K(x,z)\, K_2(z,t)dz$$

Again, using Eqs. (v) and (vi)

$$K_3(x,t) = \int_0^{\pi} e^x \cos z \cdot \left\{(-1)^1 \frac{1+e^{\pi}}{2} \cdot e^z \cdot \cos t\right\} dz$$

$$= -\frac{1+e^{\pi}}{2} \cdot e^x \cos t \cdot \int_0^{\pi} e^z \cos z dz$$

$$= -\frac{(1+e^{\pi})}{2} e^x \cos t \cdot \left\{(-1)^1 \frac{1+e^{\pi}}{2}\right\}$$

or $$K_3(x,t) = (-1)^2 \left(\frac{1+e^{\pi}}{2}\right)^2 \cdot e^x \cos t \qquad \text{(vii)}$$

Looking the form of Eqs. (v), (vi) and (vii), we write

$$K_n(x,t) = (-1)^{n-1} \left(\frac{1+e^{\pi}}{2}\right)^{n-1} e^x \cdot \cos t; \qquad n = 1, 2, 3, \ldots$$

(c) Here, with kernel $K(x,t) = x + \sin t$, we have $a = -\pi$ and $b = \pi$.

Now, using Eq. (i),

$$K_1(x,t) = K(x,t) = (x + \sin t) \qquad \ldots \text{(viii)}$$

For $n = 2$, Eq. (ii) gives

$$K_2(x,t) = \int_{-\pi}^{\pi} K(x,z) \cdot K_1(z,t)\, dz$$

or $$K_2(x,t) = \int_{-\pi}^{\pi} (x + \sin z)(z + \sin t)\, dz$$

$$K_2(x,t) = x\int_{-\pi}^{\pi} z\, dz + \sin t \int_{-\pi}^{\pi} \sin z\, dz + x \sin t \int_{-\pi}^{\pi} dz + \int_{-\pi}^{\pi} z \sin z\, dz$$

$$K_2(x,t) = x \cdot 0 + \sin t \cdot 0 + x \sin t\, (z)_{-\pi}^{\pi} + [z \cdot (-\cos z)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1 \cdot (-\cos z)\, dz$$

$$K_2(x,t) = x \sin t \cdot (2\pi) + 2\pi = 2\pi(1 + x \sin t) \qquad \text{(ix)}$$

Next, for $K_3(x,t)$, let $n = 3$ in Eq. (ii), using the previous results,

$$K_3(x,t) = \int_{-\pi}^{\pi} K(x,z) \cdot K_2(z,t) dz = \int_{-\pi}^{\pi} (x + \sin z)\{2\pi(1 + z \sin t)\} dz$$

$$K_3(x,t) = 4\pi^2 (x + \sin t) \qquad \text{(x)}$$

Further, for $K_4(x,t)$, let $n = 4$ in Eq. (ii) and using the previons results,

$$K_4(x,t) = \int_{-\pi}^{\pi} K(x,z) \cdot K_3(z,t) dz$$

$$K_4(x,t) = \int_{-\pi}^{\pi} (x + \sin z) \cdot \{4\pi^2 (z + \sin t)\} dz$$

$$K_4(x,t) = 4\pi^2 \cdot 2\pi(1 + x \sin t) \tag{xi}$$

or
$$K_4(x,t) = 4\pi^2 \cdot K_2(x,t) \tag{xii}$$

Similarly,
$$K_5(x,t) = 4\pi^2 K_3(x,t) = 16\pi^4 K_1(x,t) \tag{xiii}$$

$$K_6(x,t) = 4\pi^2 K_4(x,t) = 16\pi^4 K_2(x,t) \tag{xiv}$$

In general,

If $\quad n = 2m - 1, K_{2m-1}(x,t) = (2\pi)^{2m-2}(x + \sin t), m = 1, 2, 3, \ldots$

$\quad n = 2m, K_{2m}(x,t) = (2\pi)^{2m-1}(1 + x \sin t), m = 1, 2, 3, \ldots$

(d) Here, we have $K(x,t) = x - t$, $a = 0$ and $b = 1$.

Applying Eq. (i),

$$K_1(x,t) = K(x,t) = x - t \tag{xv}$$

and by Eq. (ii) for $n = 2$,

$$K_2(x,t) = \int_0^1 K(x,z) \cdot K_1(z,t) dz$$

or
$$K_2(x,t) = \int_0^1 (x - z)(z - t) dz \qquad \text{[By Eq. (xv)]}$$

$$K_2(x,t) = \frac{x + t}{2} - \frac{1}{3} - xt \tag{xvi}$$

Next, for $n = 3$, by Eq. (ii),

$$K_3(x,t) = \int_0^1 K(x,z) \cdot K_2(z,t) dz$$

Using Eqs. (xv) and (xvi), we have

$$K_3(x,t) = \int_0^1 (x - z) \left[\frac{z + t}{2} - \frac{1}{3} - zt \right] dz$$

After simplification,

$$K_3(x,t) = -\frac{(x - t)}{12} = -\frac{1}{12} K_1(x,t) \tag{xvii}$$

Further for $n = 4$, by Eq. (ii),

$$K_4(x,t) = \int_0^1 K(x,z) \cdot K_3(z,t) dz$$

which upon using Eqs. (xv) and (xvii) becomes

$$K_4(x,t) = \int_0^1 (x - z) \cdot \left\{ \frac{-(z - t)}{12} \right\} dz$$

Upon solving,

$$K_4(x,t) = \frac{-1}{12}\left(\frac{x+t}{2} - \frac{1}{3} - xt\right) = \frac{-1}{12}K_2(x,t) \tag{xviii}$$

Proceeding similarly, we find

$$K_5(x,t) = -\frac{1}{12}K_3(x,t) = \left(\frac{-1}{12}\right)^2 K_1(x,t)$$

and

$$K_6(x,t) = -\frac{1}{12}K_4(x,t) = \left(\frac{-1}{12}\right)^2 K_2(x,t)$$

Generalising, if $n = 2m - 1$, then

$$K_{2m-1} = \frac{(-1)^{m-1}}{12^{m-1}}(x-t), \; m = 1,2,3...$$

and if $n = 2m$, then

$$K_{2m} = \frac{(-1)^{m-1}}{12^{m-1}}\left(\frac{x+t}{2} - \frac{1}{3} - xt\right), \quad m = 1,2,3...$$

Type 2 Determination of resolvent kernel (or reciprocal kernal) R(x, t; λ)

Using Eq. (6.48), the corresponding kernel for Neumann series is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \tag{6.53}$$

The procedure can be followed by the example given hereunder.

EXAMPLE 6.2: Determine the resolvent kernel for the Fredholm integral equation having kernels:

(a) $K(x,t) = e^{x+t}$; $a = 0$, $b = 1$

(b) $K(x,t) = (1 + x)(1 - t)$; $a = 0$, $b = 1$

Solution: (a) For $K(x,t) = e^{x+t}$, $a = 0$, $b = 1$, [using Eq. (6.2)]

$$K_1(x,t) = K(x,t) = e^{x+t} \tag{i}$$

and

$$K_m(x,t) = \int_0^1 K(x,z)\cdot K_{m-1}(z,t)dz \tag{ii}$$

Let $m = 2$, So, $K_2(x,t) = \int_0^1 K(x,z)\cdot K_1(z,t)dz$

$$K_2(x,t) = \int_0^1 e^{x+z}\cdot e^{z+t}dz = e^{x+t}(e^2-1)/2 \tag{iii}$$

Let $m = 3$, So, $K_3(x,t) = \int_0^1 K(x,z)\cdot K_2(z,t)dz$

$$K_3(x,t) = \int_0^1 e^{x+z}\cdot e^{z+t}\left(\frac{e^2-1}{2}\right)dz = e^{x+t}\left(\frac{e^2-1}{2}\right)^2 \tag{iv}$$

An observation over Eqs. (iii) and (iv) leads to

$$K_m(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}, \quad m = 1,2,3,... \tag{v}$$

Thus, the required resolvent kernel $R(x,t;\lambda)$ given by Eq. (6.48) is,

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} . e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1} \quad \text{[Using Eq. (v)]}$$

$$R(x,t;\lambda) = e^{x+t}. \sum_{m=1}^{\infty}\left(\frac{\lambda(e^2-1)}{2}\right)^{m-1} \tag{vi}$$

We find $\displaystyle\sum_{m=1}^{\infty}\left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} = 1 + \frac{\lambda(e^2-1)}{2} + \left\{\frac{\lambda(e^2-1)}{2}\right\}^2 + ...$

As it is an infinite geometric series with the common ratio $\dfrac{\lambda(e^2-1)}{2}$,

so
$$\sum_{m=1}^{\infty}\left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} = \frac{1}{1-\dfrac{\lambda(e^2-1)}{2}} = \frac{2}{2-\lambda(e^2-1)} \tag{vii}$$

Provided
$$\left|\frac{\lambda(e^2-1)}{2}\right| < 1 \Rightarrow |\lambda| < \frac{2}{e^2-1} \tag{viii}$$

Finally, using Eq. (vii) in Eq. (vi), the required resolvent kernel is

$R(x,t;\lambda) = e^{x+t} \cdot \left\{\dfrac{2}{2-\lambda(e^2-1)}\right\}$, provided Eq. (viii) is satisfied.

(b) Using Eq. (6.2), for $K(x,t) = (1+x)(1-t), a = 0, b = 1$,

we have
$$K_1(x,t) = K(x,t) = (1+x)(1-t) \tag{ix}$$

and
$$K_m(x,t) = \int_0^1 K(x,z)\cdot K_{m-1}(z,t)dz$$

Let $m = 2$, So, $K_2(x,t) = \int_0^1 K(x,z)\cdot K_1(z,t)dz$

$$K_2(x,t) = \int_0^1 (1+x)(1-z)\cdot (1+z)(1-t)dz$$

or
$$= (1+x)(1-t)\int_0^1 (1-z^2)\,dz = \frac{2}{3}(1+x)(1-t) \tag{x}$$

Now, let $m = 3$, So,

$$K_3(x,t) = \int_0^1 K(x,z)\cdot K_2(z,t)dz$$

$$K_3(x,t) = \int_0^1 (1+x)(1-z)\cdot\frac{2}{3}(1+z)(1-t)dz = \left(\frac{2}{3}\right)^2 (1+x)(1-t) \tag{xi}$$

Observing Eq. (x) and (xi), we express

$$K_m(x,t) = \left(\frac{2}{3}\right)^{m-1} (1+x)(1-t) \tag{xii}$$

Now, the resolvent kernel is given by Eq. (6.48), which is [by Eq. (xii)]

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} \left\{\left(\frac{2}{3}\right)^{m-1} (1+x)(1-t)\right\}$$

$$R(x,t;\lambda) = (1+x)(1-t) \sum_{m=1}^{\infty} \left(\frac{2\lambda}{3}\right)^{m-1} \tag{xiii}$$

We find that $\sum\limits_{m=1}^{\infty} \left(\frac{2\lambda}{3}\right)^{m-1} = 1 + \frac{2\lambda}{3} + \left(\frac{2\lambda}{3}\right)^2 + \ldots.$

is an infinite G.P.* with common ratio $\frac{2\lambda}{3}$. Hence, this sum

$$\sum_{m=1}^{\infty} \left(\frac{2\lambda}{3}\right)^{m-1} = \frac{1}{1-\frac{2\lambda}{3}} = \frac{3}{3-2\lambda}, \text{ provided } \left|\frac{2\lambda}{3}\right| < 1 \text{ or } |\lambda| < \frac{3}{2} \tag{xiv}$$

Summing up, the required resolvent kernel using Eqs. (xiii) and (xiv) is

$$R(x,t;\lambda) = \frac{3(1+x)(1-t)}{3-2\lambda}, \text{ provided } |\lambda| < \frac{3}{2} \tag{xv}$$

Type 3: Solution of Fredholm integral equation with the help of resolvent kernel

Let

$$y(x) = f(x) + \lambda \int_a^b K(x,t) \cdot y(t) dt \tag{6.54}$$

be given Fredholm integral equation.

Let $K_m(x,t)$ be the m^{th} iterated kernel and let $R(x,t;\lambda)$ be the resolvent kernel for Eq. (6.54), which is given by [refer to Eq. (6.48)]

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t) \tag{6.55}$$

Now, if the sum of the infinite series [Eq. (6.55) exists, i.e., $R(x,t;\lambda)$ can be obtained in closed form (as found in Example 6.2), then the required solution of Eq. (6.54) is given by

$$y(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t)\, dt \tag{6.56}$$

---

* For G.P. $a + ar + ar^2 + \ldots\infty, S_\infty = \frac{a}{1-r}$, provided $|r| < 1$.

EXAMPLE 6.3: Solve the integral equation

$$y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 x.t \cdot y(t)dt$$

by the method of successive approximation.

Solution: The given equation is

$$y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 x \cdot t \cdot y(t)dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.54), we have

$$f(x) = \frac{5x}{6}, \lambda = \frac{1}{2}, K(x,t) = xt, a = 0, b = 1$$

The iterated kernel $K_m(x,t)$ is given by Eq. (6.2), and we have

$$K_1(x,t) = K(x,t) = xt \tag{ii}$$

and $$K_m(x,t) = \int_0^1 K(x,z) \cdot K_{m-1}(z,t)dz$$

Let m = 2. so, $$K_2(x,t) = \int_0^1 (xz)(zt)dz = \frac{1}{3}xt \tag{iii}$$

For m = 3, $$K_3(x,t) = \int_0^1 (xz)\left(\frac{1}{3}zt\right)dz = \left(\frac{1}{3}\right)^2 xt \tag{iv}$$

Thus, we have $K_m(x,t) = \left(\frac{1}{3}\right)^{m-1} xt$, m = 1, 2, 3.. (v)

Now, by Eq. (6.48), the resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(x,t)$$

In our case, for $\lambda = \frac{1}{2}$ and $k_m(x,t) = \left(\frac{1}{3}\right)^{m-1} xt$, we have

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} \left(\frac{1}{3}\right)^{m-1} xt = xt\sum_{m=1}^{\infty} \left(\frac{1}{6}\right)^{m-1}$$

$$R(x,t;\lambda) = xt\left[1 + \frac{1}{6} + \frac{1}{6^2} + \cdots\right] = xt\left(\frac{1}{1-1/6}\right) = \frac{6xt}{5}$$

Finally, the required solution of Eq. (i) is given by Eq. (6.56) as

$$y(x) = f(x) + \lambda\int_0^1 R(x,t;\lambda) \cdot f(t)dt$$

is the present case, substituting the value, we obtain

$$y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 \frac{6xt}{5} \cdot \frac{5t}{6}dt$$

$$y(x) = \frac{5x}{6} + \frac{1}{2} \cdot \frac{x}{3} = x$$

EXAMPLE 6.4: By iterative method, solve

$$y(x) = 1 + \lambda \int_0^{\pi} \sin(x+t) \cdot y(t)\, dt$$

Solution: The given equations is

$$y(x) = 1 + \lambda \int_0^{\pi} \sin(x+t) \cdot y(t)\, dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.54), we have

$$f(x) = 1, \lambda = \lambda, K(x,t) = \sin(x+t)$$

The m^{th} iterated kernel $K_m(x,t)$ following Eq. (6.2) is given by

$$K_1(x,t) = K(x,t) = \sin(x+t) \tag{ii}$$

$$K_m(x,t) = \int_0^{\pi} K(x,z) \cdot K_{m-1}(z,t) dz$$

Let $m = 2$. So, $$K_2(x,t) = \int_0^{\pi} K(x,z) \cdot K_1(z,t) dz$$

In the present case,

$$K_2(x,t) = \int_0^{\pi} \sin(x+z) \cdot \sin(z+t) dz$$

or

$$K_2(x,t) = \frac{1}{2} \int_0^{\pi} [\cos(x-t) - \cos(2z+x+t)] dz$$

$$K_2(x,t) = \frac{1}{2} \left[z \cos(x-t) - \frac{1}{2} \sin(2z+x+t) \right]_0^{\pi}$$

$$K_2(x,t) = \frac{1}{2} \left[\pi \cos(x-t) - \frac{1}{2} \sin(x+t) + \frac{1}{2} \sin(x+t) \right] = \frac{\pi}{2} \cos(x-t) \tag{iii}$$

Now, let $m = 3$, So,

$$K_3(x,t) = \int_0^{\pi} K(x,z) \cdot K_2(z,t) dz$$

Putting for $K(x,z)$, and $K_2(x,t)$

$$K_3(x,t) = \int_0^{\pi} \sin(x+z) \cdot \frac{\pi}{2} \cos(z-t) dz$$

which simplifies to $$K_3(x,t) = \left(\frac{\pi}{2} \right)^2 \sin(x+t) \tag{iv}$$

By $m = 4$, $$K_4(x,t) = \left(\frac{\pi}{2} \right)^3 \cos(x-t)$$

and by $m = 5$, $$K_5(x,t) = \left(\frac{\pi}{2} \right)^4 \cdot \sin(x+t) \tag{v}$$

Now, looking at Eq. (ii) to (v), the symmetry is among odd and even m, and we express the resolvent kernel $R(x,t;\lambda)$ by

$$R(x,t;\lambda)=\sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t)$$

$$R(x,t;\lambda)=K_1(x,t)+\lambda K_2(x,t)+\lambda^2K_3(x,t)+\cdots$$

$$R(x,t;\lambda)=\{K_1(x,t)+\lambda^2K_3(x,t)+\lambda^4K_5(x,t)+\cdots\}$$

$$+\lambda\{K_2(x,t)+\lambda^3K_4(x,t)+\lambda^5K_6(x,t)+\cdots\}$$

$$R(x,t;\lambda)=\sin(x+t)\cdot\left\{1+\left(\frac{\lambda\pi}{2}\right)^2+\left(\frac{\lambda\pi}{2}\right)^4+\cdots\right\}$$

$$+\frac{\lambda\pi}{2}\cdot\cos(x-t).\left\{1+\left(\frac{\lambda\pi}{2}\right)^2+\left(\frac{\lambda\pi}{2}\right)^4+\cdots\right\}$$

$$R(x,t;\lambda)=\left\{\sin(x+t)+\frac{\lambda\pi}{2}\cos(x-t)\right\}\left\{\frac{1}{1-(\lambda\pi/2)^2}\right\},\text{ provided }\left(\frac{\lambda\pi}{2}\right)^2<1$$

$$R(x,t;\lambda)=\frac{2}{4-\lambda^2\pi^2}[2\sin(x+t)+\lambda\pi\cos(x-t)] \tag{vi}$$

Finally, the required solution is given by [refer to Eq. (6.56)]

$$y(x)=f(x)+\lambda\int_0^{\pi}R(x,t;\lambda)\cdot f(t)dt$$

$$y(x)=1+\frac{2\lambda}{4-\lambda^2\pi^2}\int_0^{\pi}[2\sin(x+t)+\lambda\pi\cos(x-t)]dt$$

$$y(x)=1+\frac{4\lambda}{4-\lambda^2\pi^2}(2\cos x+\lambda\pi\sin x),|\lambda|<\frac{2}{\pi}\quad\text{[After simplification]}$$

EXERCISE 6.1

1. $y(x)=x+\int_0^{1/2}y(t)\,dt$

2. $y(x)=e^x-\frac{e}{2}+\frac{1}{2}+\frac{1}{2}\int_0^1 y(t)\,dt$

3. $y(x)=x+\lambda\int_0^1 xt\cdot y(t)\,dt$

4. $y(x)=\sin x-\frac{x}{4}+\frac{1}{4}\int_0^{\pi/2}xt\cdot y(t)\,dt$

5. $y(x)=f(x)+\lambda\int_0^1 e^{x-t}\cdot y(t)\,dt$

6. $y(x)=\frac{3}{2}e^x-\frac{1}{2}xe^x-\frac{1}{2}+\frac{1}{2}\int_0^1 t\cdot y(t)\,dt$

7. $y(x)=1+\lambda\int_0^1(1-3xt)\cdot y(t)\,dt$

Answers:

1. $x+\frac{1}{4}$
2. e^x
3. $(3x)/(3-\lambda), |\lambda|<3$
4. $\sin x$.
5. $f(x)+\frac{\lambda}{1-\lambda}\int_0^1 e^{x-t}f(t)dt, |\lambda|<1$
6. $\frac{3}{2}e^x-\frac{1}{2}xe^x-\frac{e}{3}+1$
7. $\frac{4+2\lambda(2-3x)}{4-\lambda^2}, |\lambda|<2$

Type 4: Solution of Fredholm integral equation when the resolvent kernel cannot be obtained in closed form

In the case when the infinite series occuring in the formulae of the resolvent kernel cannot be determined, we use the method of successive approximations to find the solution upto the third order.

Let the given Fredholm integral equation of the second kind be

$$y(x)=f(x)+\lambda\int_a^b K(x,t)\cdot y(t)dt \tag{6.57}$$

First, we take zero order approximation as

$$y_0(x)=f(x) \tag{6.58}$$

Now, if the n^{th} order approximation is denoted by $y_n(x)$, we express

$$y_n(x)=f(x)+\lambda\int_a^b K(x,t)\cdot y_{n-1}(t)dt \tag{6.59}$$

Then, with the help of Eqs. (6.58) and (6.59), we find $y_1(x)$, $y_2(x)$ and $y_3(x)$.

Note: In case zero order approximation is provided with the problem, we modify our results, i.e., Eqs. (6.58) and (6.59).

EXAMPLE 6.5: Solve the inhomogeneous Fredholm integral equation of the second kind

$$y(x)=2x+\lambda\int_0^1(x+t)\cdot y(t)dt$$

by the method of successive approximations to the third order by taking $y_0(x)=1$.

Solution:

The given equation is

$$y(x) = 2x + \lambda\int_0^1 (x+t)\cdot y(t)dt \qquad \text{(i)}$$

The zero-order approximation is

$$y_0(x) = 1 \qquad \text{(ii)}$$

If $y_n(x)$ is n^{th} order approximation, then we apply Eq. (6.59), by which

$$y_n(x) = 2x + \lambda\int_0^1 (x+t)\,y_{n-1}(t)dt \qquad \text{(iii)}$$

Let $n = 1$; $$y_1(x) = 2x + \lambda\int_0^1 (x+t)\cdot 1\,dt \qquad \text{[using Eq. (ii)]}$$

or $$y_1(x) = 2x + \lambda\left(x + \frac{1}{2}\right)$$

Next, let $n = 2$. Then, by Eq. (iii),

$$y_2(x) = 2x + \lambda\int_0^1 (x+t)\cdot y_1(t)\,dt$$

$$y_2(x) = 2x + \lambda\int_0^1 (x+t)\left\{2t + \lambda\left(t + \frac{1}{2}\right)\right\}dt\ ; \qquad \text{[By Eq. (iv)]}$$

or $$y_2(x) = 2x + \lambda\left(x + \frac{2}{3}\right) + \lambda^2\left(x + \frac{7}{12}\right), \qquad \text{[After simplification] (v)}$$

Next, let $n = 3$. Then, by Eq. (iii),

$$y_3(x) = 2x + \lambda\int_0^1 (x+t)\cdot y_2(t)dt$$

$$y_3(x) = 2x + \lambda\int_0^1 (x+t)\left\{2t + \lambda\left(t + \frac{2}{3}\right) + \lambda^2\left(t + \frac{7}{12}\right)\right\}dt$$

or $$y_3(x) = 2x + \lambda\left(x + \frac{2}{3}\right) + \lambda^2\left(\frac{7x}{6} + \frac{2}{3}\right) + \lambda^3\left(\frac{13x}{12} + \frac{5}{8}\right) \qquad \text{(vi)}$$

EXERCISES 6.2

1. $y(x) = 1 + \lambda\int_0^1 (x+t)\cdot y(t)dt$

Answer:

1. $y_3(n) = 1 + \lambda\left(x + \frac{1}{2}\right) + \lambda^2\left(x + \frac{7}{12}\right) + \lambda^3\left(\frac{13x}{12} + \frac{5}{8}\right)$

6.9 RECIPROCAL FUNCTIONS

The iterated kernel $K_n(x,t)$, $n = 1,2,3, \ldots$ is given by

$$K_1(x,t) = K(x,t) \qquad [6.60(a)]$$

and $$K_n(x,t) = \int_a^b K(x,z)\cdot K_{n-1}(z,t)dz \qquad [6.60(b)]$$

Now, let $$-k(x,t) = K_1(x,t) + K_2(x,t) + \cdots + K_n(x,t) + \cdots \qquad (6.61)$$

Here, $K(x,t)$ is real and continuous in a rectangle R, for which $a \le x \le b$ $a \le t \le b$. Let $K(x,t) \not\equiv 0$ and let M be the maximum value of $|K(x,t)|$ in R, meaning by $|K(x,t)| \le M$ in R. Then, if $M(b - a) < 1$, it is found that the infinite series Eq. (6.61), for $k(x, t)$ is absolutely and uniformly convergent. Hence, $k(x,t)$ is real and continuous in R.

Now, (after going through Chapter 5), we know that

$$K_{p+q}(x,t) = \int_a^b K_p(x,z)\cdot K_q(z,t)dz \qquad (6.62)$$

Then, Eq. (6.61) can be expressed as

$$-k(x,t) - K_1(x,t) = K_2(x,t) + K_3(x,t) + \cdots + K_n(x,t) + \cdots$$

For $K_1(x,t)$, we use Eq. [6.60(a)] and express this as [using Eq. (6.62)]

$$\begin{aligned} -k(x,t) - K(x,t) &= K_2(x,t) + K_3(x,t) + \cdots + K_n(x,t) + \cdots \\ &= \int_a^b K_1(x,z).K_1(z,t)dz + \int_a^b K_1(x,z)\cdot K_2(z,t)dz + \cdots \\ &= \int_a^b K_1(x,z)[K_1(z,t) + K_2(z,t) + \cdots]dz \end{aligned} \qquad (6.63)$$

which upon using Eqs. [6.60(a)] and (6.61), becomes

$$-k(x,t) - K(x,t) = -\int_a^b K(x,z)\cdot k(z,t)dz \qquad (6.64)$$

Again, using Eq. (6.62), Eq. (6.63) may be expressed as

$$\begin{aligned} -k(x,t) - K(x,t) &= \int_a^b K_1(x,z).K_1(z,t)dz + \int_a^b K_2(x,z).K_1(z,t)dz + \cdots. \\ &= \int_a^b [K_1(x,z) + K_2(x,z) + \cdots]\, K_1(z,t)dz \end{aligned}$$

which upon using Eqs. [6.60(a)] and (6.61), becomes

$$-k(x,t) - K(x,t) = -\int_a^b k(x.z)\cdot K(z,t)dz \qquad (6.65)$$

Now, from Eqs. (6.64) and (6.65), we have

$$k(x,t) + K(x,t) = \int_a^b K(x,z)\cdot k(z,t)dz \qquad [6.66(a)]$$

and $$k(x,t) + K(x,t) = \int_a^b k(x,z)\cdot K(z,t)dz \qquad [6.66(b)]$$

Two functions $K(x,t)$ and $k(x,t)$ are said to be reciprocal if they both are real and continuous in R and satisfy the conditions given in Eq. [6.66(a)] or Eq. [6.66(b)].

Theorem: If $K(x,t)$ is real and continuous in R, there exists a reciprocal function $k(x,t)$ given by

$$-k(x,t) = K_1(x,t) + K_2(x,t) + \cdots + K_n(x,t) \tag{6.67}$$

where $K_1(x,t)$, $K_2(x,t)$,... are iterated functions (or kernels), provided that $M(b - a) < 1$, where M is the maximum value of $|K(x,t)|$ in R for which $a \le x \le b$ $a \le t \le b$.

(Proof of this theorem has already been covered under section 6.9)

6.10 ANOTHER APPROACH TO SOLVE FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND (VOLTERRA SOLUTION)

Therem: Let the Fredholm integral equation* be

Now, if
$$y(x) = f(x) + \int_a^b K(x,t)\cdot y(t)dt \tag{6.68}$$

1. $K(x,t)$ is real and continuous in R, for which $a \le x \le b$, and $a \le t \le b$, $K(x,t) \not\equiv 0$;
2. $f(x)$ is real and continuous in I $(a \le x \le b)$ and $f(x) \not\equiv 0$;
3. a function $k(x,t)$ reciprocal to $K(x,t)$ exists;

then the integral equation, i.e., Eq. (6.68) has a unique continuous solution in I given by

$$y(x) = f(x) - \int_a^b k(x,t)\cdot f(t)dt \tag{6.69}$$

Proof: Let the variable of integration be z in place of t in Eq. (6.68).
Then, replacing x by t, we have

$$y(t) = f(t) + \int_a^b K(t,z)\cdot y(z)dz \tag{6.70}$$

Multiplying both sides by $k(x,t)$ and then integrating with respect to t $(a \le t \le b)$, we have

$$\int_a^b k(x,t).y(t)dt = \int_a^b k(x,t)\cdot f(t)dt + \int_a^b k(x,t)\left\{\int_a^b K(t,z)\cdot y(z)dz\right\}dt$$

In the last integral interchanging the order of integration, we get

$$\int_a^b k(x,t)\cdot y(t)dt = \int_a^b k(x,t)\cdot f(t)dt + \int_a^b y(z)\left\{\int_a^b k(x,t)\cdot K(t,z)dt\right\}dz \tag{6.71}$$

Now, since $k(x,t)$ and $K(x,t)$ are reciprocal functions, we have [by (6.66)b]

$$\int_a^b k(x,t).K(t,z)dt = k(x,z) + K(x,z) \tag{6.72}$$

* Note that here, λ is merged with $K(x,t)$.

Using Eq. (6.72) in Eq. (6.71), we get

$$\int_a^b k(x,t).y(t)dt = \int_a^b k(x,t)\cdot f(t)dt + \int_a^b y(z)\{k(x,z)+K(x,z)\}dz$$

or, $$\int_a^b k(x,z)\cdot y(z)dz = \int_a^b k(x,t)\cdot f(t)dt + \int_a^b k(x,z)\cdot y(z)dz + \int_a^b K(x,z)\cdot y(z)dz$$

or $$0 = \int_a^b k(x,t)\cdot f(t)dt + \int_a^b K(x,t)\cdot y(t)\,dt \tag{6.73}$$

Now, from (6.69),

$$\int_a^b K(x,t)\cdot y(t)dt = y(x) - f(x) \tag{6.74}$$

Using Eq. (6.74), Eq. (6.72) can be written as

$$y(x) = f(x) - \int_a^b k(x,t)\,f(t)dt \tag{6.75}$$

It means that if Eq. (6.68) has a continuous solution, then it is given by Eq. (6.75), and it is unique also.

EXAMPLE 6.6: Solve $y(x) = f(x) + \frac{1}{2}\int_0^1 e^{x-t}\cdot y(t)\,dt$ using Volterra method.

Solution: The given equation is

$$y(x) = f(x) + \frac{1}{2}\int_0^1 e^{x-t}\cdot y(t)\,dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.68), we have

$$K(x,t) = \frac{1}{2}e^{x-t} \tag{ii}$$

Let $k(x,t)$ be reciprocal kernel of $K(x,t)$. Then, if $K_1(x,t)$, $K_2(x,t)$,... are iterated functions, then by Eq. (6.67),

$$-k(x,t) = K_1(x,t) + K_2(x,t) + \cdots \tag{iii}$$

And these iterated kernels are defined by Eq. (6.2), presently

$$K_1(x,t) = K(x,t) = \frac{1}{2}e^{x-t} \tag{iv}$$

and $$K_n(x,t) = \int_0^1 K(x,z)\cdot K_{n-1}(z,t)dz, \qquad n = 2,3,... \tag{v}$$

For $n = 2$, $$K_2(x,t) = \int_0^1 K(x,z)\cdot K_1(z,t)dz$$

$$K_2(x,t) = \int_0^1 \frac{1}{2}e^{x-z}\cdot\frac{1}{2}e^{z-t}\,dz = \frac{1}{2^2}e^{x-t} \tag{vi}$$

For $n = 3$ $$K_3(x,t) = \int_0^1 K(x,z)\cdot K_2(z,t)dz$$

$$K_3(x,t) = \int_0^1 \frac{1}{2}e^{x-z}\cdot\frac{1}{2^2}e^{z-t}\,dz = \frac{1}{2^3}e^{x-t} \tag{vii}$$

Looking at Eq. (v), (vi) and (vii), we can express

$$K_n(x,t) = \frac{1}{2^n} e^{x-t}, \quad n = 1,2,3,... \tag{viii}$$

Substituting the values of $K_1(x,t)$, $K_2(x,t)$,... in Eq. (iii), we have

$$-k(x,t) = (e^{x-t})\left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right] = \frac{1}{2}.e^{x-t} \cdot \frac{1}{1-\frac{1}{2}} = e^{x-t}$$

So, $$k(x,t) = -e^{x-t} \tag{ix}$$

Now, by Eq. (6.75), the solution of Eq. (i) is given by

$$y(x) = f(x) + \int_0^1 e^{x-t} \cdot f(t)\,dt \tag{x}$$

and this is the required solution.

Note: The practice questions given in Exercise 6.1 can also be solved by Volterra method.

6.11 SOLUTION OF VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND BY SUCCESSIVE APPROXIMATIONS: ITERATIVE METHOD (NEUMANN SERIES)

Let us consider the following Volterra integral equation of the second kind:

$$y(x) = f(x) + \lambda \int_a^x K(x,t) \cdot y(t)\,dt \tag{6.76}$$

and let the zero order approximation for the required solution $y(x)$ be

$$y_0(x) = f(x) \tag{6.77}$$

Further, let $y_n(x)$ and $y_{n-1}(x)$ be nth and $(n-1)$th order approximations and these are connected by

$$y_n(x) = f(x) + \lambda \int_a^x K(x,t) \cdot y_{n-1}(t)\,dt \tag{6.78}$$

We also define [refer to Eqs. [6.4(a)] and [6.4(b)] the iterated kernels as

$$K_1(x,t) = K(x,t) \qquad [(6.79(a)]$$

and $$K_n(x,t) = \int_t^x K(x,z) \cdot K_{n-1}(z,t)dz \qquad [6.79(b)]$$

Now, by Eq. (6.78), for $n = 1$,

$$y_1(x) = f(x) + \lambda \int_a^x K(x,t)\, y_0(t)dt$$

$$y_1(x) = f(x) + \lambda \int_a^x K(x,t)\ .f(t)dt \quad \text{[using Eq. (6.77)]} \tag{6.80}$$

or $$y_1(z) = f(z) + \lambda\int_a^z K(z,t)\cdot f(t)dt \text{ [Replacing } x \text{ by } z] \tag{6.81}$$

Again, by Eq. (6.78), for $n = 2$,

$$y_2(x) = f(x) + \lambda\int_a^x K(x,t)\cdot y_1(t)dt$$

or $$y_2(x) = f(x) + \lambda\int_a^x K(x,z)\cdot y_1(z)dz \text{ [By replacing } t \text{ by } z] \tag{6.82}$$

Now, substituting $y_1(z)$ from Eq. (6.81), Eq. (6.82) becomes

$$y_2(x) = f(x) + \lambda\int_a^x K(x,z)\left[f(z) + \lambda\int_a^z K(z,t).f(t)dt\right]dz$$

or $$y_2(x) = f(x) + \lambda\int_a^x K(x,z).f(z)\,dz + \lambda^2\int_{z=a}^x K(x,z)\left[\int_{t=a}^z K(z,t).f(t)dt\right]dz \tag{6.83}$$

We now change the order of integration. Figure [6.2(a)] shows the last double integral of Eq. (6.83). For changing the order, we refer Figure [6.2(b)] for the same area of integration.

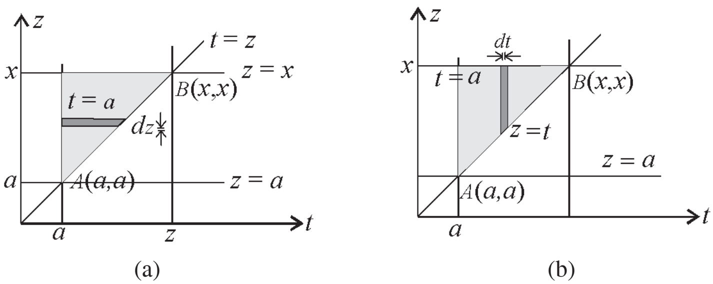

Figure 6.2 Supporting diagrams for change of order of integration.

or, $$y_2(x) = f(x) + \lambda\int_a^x K(x,z).f(z)dz + \lambda^2\int_a^x f(t)\left[\int_{z=t}^x K(x,z).K(z,t)dz\right]dt \tag{6.84}$$

Now, by [6.79(b) for $n = 2$,

$$K_2(x,t) = \int_t^x K(x,z)\cdot K_1(z,t)dz = \int_t^x K(x,z)\cdot K(z,t)dz \quad \text{[Using (6.79(a)]}$$

Now, using this result of $K_2(x,t)$ in R.H.S. of Eq. (6.84) for double integral part, we have [replacing z by t in the second integral, along with Eq. [6.79(a)]

$$y_2(x) = f(x) + \lambda\int_a^x K_1(x,t).f(t)dt + \lambda^2\int_a^x f(t).K_2(x,t)dt$$

or $$y_2(x) = f(x) + \sum_{m=1}^{2}\lambda^m\int_a^x K_m(x,t).f(t)dt \tag{6.85}$$

Proceeding similarly, we can express Eq. (6.85) as

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^x K_m(x,t).f(t)dt \tag{6.86}$$

Now, taking the limit $n \to \infty$, we get the Neumann Series.

$$y(x) = \lim_{n\to\infty} y_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x,t).f(t)dt \tag{6.87}$$

We now determine the resolvent kernel $R(x,t;\lambda)$ in terms of iterated kernel $K_m(x,t)$. Changing the order of summation and integration of Eq. (6.87), we obtain

$$y(x) = f(x) + \lambda \int_a^x \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \right].f(t)dt \tag{6.88}$$

Now, comparing Eq. (6.88) with Eq. [6.6(a)], we have

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \tag{6.89}$$

Equation (6.89) converges uniformly and absolutely when $K(x,t)$ is continuous in R.

6.12 RESOLVENT KERNEL AND VOLTERRA INTEGRAL EQUATION

Theorom: Let $R(x,t;\lambda)$ be the resolvent (or reciprocal) kernel of the Volterra integral equation.

$$y(x) = f(x) + \lambda \int_a^x K(x,t).y(t)\,dt \tag{6.90}$$

then
$$R(x,t;\lambda) = K(x,t) + \lambda \int_t^x K(x,z).R(z,t;\lambda)dz \tag{6.91}$$

Proof: We know that $R(x,t;\lambda)$ is given by (refer to section 6.11)

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \tag{6.92}$$

where, the iterated kernels are given by

$$K_1(x,t) = K(x,t) \tag{6.93(a)}$$

and
$$K_m(x,t) = \int_t^x K(x,z).K_{m-1}(z,t)dz \tag{6.93(b)}$$

Now, from Eq. (6.89), we have

$$R(x,t;\lambda) = K_1(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} K_m(x,t)$$

For R.H.S., using Eqs. [6.93(a)] and [6.93(b)],

$$R(x,t;\lambda) = K(x,t) + \sum_{m=2}^{\infty} \lambda^{m-1} \left\{ \int_t^x K(x,z).K_{m-1}(z,t)dz \right\}$$

Now, let $m-1=n$. So,

$$R(x,t;\lambda)=K(x,t)+\sum_{n=1}^{\infty}\lambda^{n}\left\{\int_t^x K(x,z).K_n(z,t)dz\right\}$$

$$R(x,t;\lambda)=K(x,t)+\lambda\sum_{m=1}^{\infty}\lambda^{m-1}\int_t^x K(x,z).K_m(z,t)dz$$

Now, changing the order of summation and integration,

$$R(x,t;\lambda)=K(x,t)+\lambda\int_t^x\left\{\sum_{m=1}^{\infty}\lambda^{m-1}K_m(z,t)\right\}K(x,z)dz$$

Finally, using Eq. (6.92), we express as

$$R(x,t;\lambda)=K(x,t)+\lambda\int_t^x K(x,z).R(z,t;\lambda)dz \tag{6.94}$$

6.13 ILLUSTRATIONS TO EXPLAIN THE SOLUTION OF VOLTERRA INTEGRAL EQUATION BY SUCCESSIVE APPROXIMATIONS (OR ITERATIVE METHOD)

Type 1: Determination of resolvent kernel of the Volterra integral equation.

The equation is $$y(x)=f(x)+\lambda\int_a^x K(x,t).y(t)\,dt \tag{6.95}$$

EXAMPLE 6.7: Find the resolvent kernel of the Volterra integral equation with kernel $K(x,t)=(2+\cos x)/(2+\cos t)$.

Solution: We know that the iterated kernels are given by

$$K_1(x,t)=K(x,t) \tag{i}$$

$$K_n(x,t)=\int_t^x K(x,z).K_{n-1}(z,t)dz\,,\quad n=2,3, \tag{ii}$$

Here, given that $K(x,t)=(2+\cos x)/(2+\cos t)$ (iii)
So, by Eq. (i) and (iii)

$$K_1(x,t)=K(x,t)=(2+\cos x)/(2+\cos t) \tag{iv}$$

For $K_2(x,t)$, let $n=2$ in Eq. (ii) , we have

$$K_2(x,t)=\int_t^x K(x,z).K_1(z,t)dz$$

Now, using Eq. (iv)

$$K_2(x,t)=\int_t^x \frac{2+\cos x}{2+\cos z}.\frac{2+\cos z}{2+\cos t}dz=\frac{2+\cos x}{2+\cos t}(x-t) \tag{v}$$

Now, let $n=3$ in Eq. (ii),

$$K_3(x,t)=\int_t^x K(x,z).K_2(z,t)dz$$

or $K_3(x,t) = \int_t^x \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t} \cdot (z-t)dz$ [Here, we have used Eq. (iii), and (v)]

$$K_3(x,t) = \frac{2+\cos x}{2+\cos t}\left[\frac{(z-t)^2}{2}\right]_t^x = \frac{2+\cos x}{2+\cos t}\,\frac{(x-t)^2}{2!} \tag{vi}$$

Similarly, by $n = 4$, we find

$$K_4(x,t) = \frac{2+\cos x}{2+\cos t}\,\frac{(x-t)^3}{3!} \tag{vii}$$

After observing the values of $K_1(x,t)$, $K_2(x,t)$, $K_3(x,t)$, $K_4(x,t)$, we mention

$$K_n(x,t) = \frac{2+\cos x}{2+\cos t}\,\frac{(x-t)^{n-1}}{(n-1)!}, \; n = 1,2,3,... \tag{viii}$$

Now, by Eq. (6.92), the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \tag{ix}$$

Expanding the R.H.S. and substituting the values of iterated kernals, we obtain

$$R(x,t;\lambda) = \left(\frac{2+\cos x}{2+\cos t}\right)\left[1 + \frac{\lambda(x-t)}{1!} + \frac{\lambda^2(x-t)^2}{2!} + \frac{\lambda^3(x-t)^3}{3!} + \cdots\right]$$

$R(x,t;\lambda) = \frac{2+\cos x}{2+\cos t}.e^{\lambda(x-t)}$. This is the required resolvent kernel.

Type 2: Solution of Volterra integral equation with the help of resolvent kernel

Let the Volterra integral equation be

$$y(x) = f(x) + \lambda\int_a^x K(x,t).y(t)dt \tag{6.96}$$

For its kernel $K(x,t)$, let $K_m(xt)$ be the iterated kernels as already given in Eqs. [6.93(a)] and [6.93(b)]. We have also learnt the method of finding the corresponding resolvent kernel in Type 1.

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \tag{6.97}$$

Let the sum of infinite series [Eq. (6.97)] be found in closed form. Then, the required solution of Eq. (6.96) is given by {also refer Section 6.11}

$$y(x) = f(x) + \lambda\int_a^x R(x,t;\lambda).f(t)dt \tag{6.98}$$

EXAMPLE 6.8: By means of resolvent kernel, find the solution of

$$y(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t}\cdot y(t)dt$$

Solution: The given equation is

$$y(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t} . y(t) dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.96), we have

$$f(x) = e^x \sin x, \lambda = 1, K(x,t) = \frac{2+\cos x}{2+\cos t} \tag{ii}$$

Now, we refer to Example 6.7, and we find the resolvent kernel as

$$R(x,t;\lambda) = \frac{2+\cos x}{2+\cos t} . e^{x-t} \tag{iii}$$

Then, the required solution is given by Eq. (6.98), which follows:

$$y(x) = e^x \sin x + 1 . \int_0^x \frac{2+\cos x}{2+\cos t} e^{x-t} . e^t \sin t \, dt \qquad \text{[by Eq. (ii) and Eq. (iii)]}$$

$$y(x) = e^x . \sin x - (2+\cos x) e^x . \int_0^x \frac{-\sin t}{2+\cos t} dt$$

$$y(x) = e^x \sin x - (2+\cos x) e^x . [\log(2+\cos t)]_0^x$$

$$y(x) = e^x \sin x - (2+\cos x) e^x . [\log(2+\cos x) - \log 3]$$

or $\quad y(x) = e^x \left[\sin x + (2+\cos x) . \log\left\{ \dfrac{3}{2+\cos x} \right\} \right]$

This is the required solution.

EXAMPLE 6.9: With the help of resolvent kernel, find the solution of the following integral equation:

$$y(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2} . y(t) dt$$

Solution:

The given equation is

$$y(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2} . y(t) dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.96), we have

$$f(x) = 1 + x^2, \lambda = 1, K(x,t) = (1+x^2)/(1+t^2) \tag{ii}$$

Using Eqs. [6.93(a)] and [6.93(b)], the iterated kernels are:

$$K_1(x,t) = K(x,t) = (1+x^2)/(1+t^2) \tag{iii}$$

and

$$K_m(x,t) = \int_t^x K(x,z) . K_{m-1}(z,t) dz \tag{iv}$$

Using Eqs. (ii) and (iii),

$$K_2(x,t) = \int_t^x \frac{1+x^2}{1+z^2} . \frac{1+z^2}{1+t^2} dz = \frac{1+x^2}{1+t^2}(x-t) \tag{v}$$

Similarly, for $m = 3$, $K_3(x,t) = \int_t^x K(x,z).K_2(z,t)dz$

or $$K_3(x,t) = \int_t^x \frac{1+x^2}{1+z^2} \cdot \frac{1+z^2}{1+t^2}(z-t)dz = \frac{1+x^2}{1+t^2}\left[\frac{(z-t)^2}{2}\right]_t^x$$

$$K_3(x,t) = \frac{1+x^2}{1+t^2} \cdot \frac{(x-t)^2}{2!} \tag{vi}$$

Similarly, for $m = 4, K_4(x,t) = \dfrac{1+x^2}{1+t^2} \cdot \dfrac{(x-t)^3}{3!}$ (vii)

Through an observation over $K_1(x,t)$, $K_2(x,t)$, ..., we express

$$K_m(x,t) = \frac{1+x^2}{1+t^2} \cdot \frac{(x-t)^{m-1}}{(m-1)!}, \quad m = 1,2,3,... \tag{viii}$$

Now, the resolvent kernel $R(x,t;\lambda)$ as given by Eq. (6.92) is

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

Expanding R.H.S. for $\lambda = 1$, and substituting the values of iterated kernels, we have

$$R(x,t;\lambda) = \frac{1+x^2}{1+t^2}\left[1 + \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} + \frac{(x-t)^3}{3!} + \cdots\right]$$

$$R(x,t;\lambda) = \frac{1+x^2}{1+t^2} e^{(x-t)} \tag{ix}$$

Finally, the required solution is given by Eq. (6.98), which in the present case is

$$y(x) = f(x) + \lambda \int_0^x R(x,t;\lambda).f(t)dt$$

$$y(x) = (1+x^2) + 1.\int_0^x \frac{1+x^2}{1+t^2}.\, e^{(x-t)}.(1+t^2)dt$$

or $$y(x) = e^x(1+x^2) \text{ [After simplication].}$$

Type 3: Solution of Volterra integral equation when the sum of the infinite series occuring in the formula for resolvent kernel cannot be found

Let the Volterra integral equation be

$$y(x) = f(x) + \lambda \int_0^x K(x,t).y(t)dt \tag{6.99}$$

With the problems mentioned above, we use the following formula, known as *Neumann series:*

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x,t).f(t)dt \tag{6.100}$$

where, $K_m(x,t)$ is the mth iterated kernel.

EXAMPLE 6.10: Find the Neumann series for the solution of integral equation:

$$y(x) = 1 + x + \lambda\int_0^x (x-t).y(t)dt$$

Solution:

The given equation is

$$y(x) = 1 + x + \lambda\int_0^x (x-t).y(t)dt \tag{i}$$

Comparing with Eq. 6.99),

$$f(x) = 1 + x,\ \lambda = \lambda,\ K(x,t) = (x - t).$$

Also, let $K_m(x,t)$ be the mth iterated kernel, then by Eqs. [(6.93(a)] and [6.93(b)],

$$K_1(x,t) = K(x,t) = x - t \tag{ii}$$

and
$$K_m(x,t) = \int_t^x K(x,z).K_{m-1}(z,t)dz \tag{iii}$$

Let $m = 2$, So, $K_2(x,t) = \int_t^x K(x,z).K_1(z,t)dz$

$$K_2(x,t) = \int_t^x (x-z)(z-t)dz \qquad \text{[by Eq. (ii)]}$$

$$K_2(x,t) = \left[(x-z).\frac{(z-t)^2}{2}\right]_t^x - \int_t^x \frac{(-1)(z-t)^2}{2}dz = 0 + \frac{(x-t)^3}{3!} \tag{iv}$$

Let $m = 3$ in Eq. (iii), we get

$$K_3(x,t) = \int_t^x K(x,z).K_2(z,t)dz$$

or
$$K_3(x,t) = \int_t^x (x-z)\frac{(z-t)^3}{3!}dz = \frac{(x-t)^5}{5!} \tag{v}$$

and so on. Now, the Neumann series [shown in Eq. (6.100) is

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x,t).f(t)dt$$

$$y(x) = (1+x) + \lambda\int_0^x K_1(x,t).(1+t)dt + \lambda^2\int_0^x K_2(x,t).(1+t)dt + \cdots$$

$$y(x) = (1+x) + \lambda\int_0^x (x-t)\ (1+t)dt + \lambda^2\int_0^x \frac{(x-t)^3}{3!}\ (1+t)dt + \cdots$$

or
$$y(x) = 1 + x + \lambda\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \lambda^2\left(\frac{x^4}{4!} + \frac{x^5}{5!}\right) + \cdots \tag{vi}$$

It is obtained after simplication, and this is the required solution of Eq. (i).

Remarks: 1. In this question, if $\lambda = 1$, then Eq. (vi) can be summed equal to e^x.
2. In this question, if we find the resolvent kernel, then

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \cdots$$

$$R(x,t;\lambda) = (x-t) + \lambda \frac{(x-t)^3}{3!} + \lambda^2 \frac{(x-t)^5}{5!} + \cdots$$

Since the sum of this series cannot be found, therefore we cannot go like Type 2.

Another example is given here in which we have adopted the procedure of Type 3.

EXAMPLE 6.11: Solve the Volterra integral equation:

$$y(x) = 1 + \int_0^x xt\,.y(t)dt$$

Solution: The given equations is

$$y(x) = 1 + \int_0^x xt\,.y(t)dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.99), we get

$$f(x) = 1, \lambda = 1, K(x,t) = x.t \tag{ii}$$

The mth iterated kernel, $K_1(x,t) = K(x,t) = xt$ (iii)

and
$$K_m(x,t) = \int_t^x K(x,z).K_{m-1}(z,t)dz \tag{iv}$$

For $K_2(x,t)$, let $m = 2$ in Eq. (iv), we get

$$K_2(x,t) = \int_t^x (xz)(zt)dz = \frac{1}{3}(x^4 t - xt^4) \tag{v}$$

For $K_3(x,t)$ we take $m = 3$ in Eq. (iv), and find

$$K_3(x,t) = \int_t^x K(x,z).K_2(z,t)dz$$

$$K_3(x,t) = \int_t^x (x.z).\frac{1}{3}(z^4 t - zt^4)\,dz = \frac{1}{18}(x^7 t - 2x^4 t^4 + xt^7) \tag{vi}$$

Similarly,
$$K_4(x,t) = \frac{1}{162}(x^{10} t - 3x^7 t^4 + 3x^4 t^7 - xt^{10}) \tag{vii}$$
and so on.

Now, it can be verified that for the pattern of iterated kernels, the sum of the series for the resolvent kernel cannot be ascertained; hence, we use Neumann's series, which is given by Eq. (6.100)

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x,t).f(t)dt \tag{viii}$$

Now, expanding for m, putting the values of $K_1(x,t)$, $K_2(x,t)$, ... as found above, and simplifying, we get

$$y(x) = 1 + \frac{x^3}{2} + \frac{x^6}{2.5} + \frac{x^9}{2.5.8} + \cdots \tag{ix}$$

This is the solution of Eq. (i).

Type 4: Method of successive approximation for solving Volterra integral equation of the second kind

Let the integral equation be

$$y(x) = f(x) + \lambda \int_0^x K(x,t).y(t)dt \tag{6.101}$$

Also, let $f(x)$ be continuous is $[0, a]$ and $K(x,t)$ be continuous for $0 \le x \le a$, $0 \le t \le x$.

We begin with some given function $y_0(x)$ continuous in $[0, a]$. Then, replacing $y(t)$ on R.H.S. of Eq. (6.101) by $y_0(x)$, we get

$$y_1(x) = f(x) + \lambda \int_0^x K(x,t).y_0(t)dt \tag{6.102}$$

Since $y_1(x)$ given by Eq. (6.102) is itself continuous in $[0, a]$, we proceed similarly and arrive at a sequence of functions $y_0(x)$, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$, ...

where,
$$y_n(x) = f(x) + \lambda \int_0^x K(x,t).y_{n-1}(t)\,dt \tag{6.103}$$

Because of continuity of $f(x)$ and $K(x,t)$, the sequence $\{y_n(x)\}$ converges as $n \to \infty$, and thus, the solution $y(x)$ is obtained.

Remark: If we take $y_0(x) = f(x)$, we obtain Neumann series.

EXAMPLE 6.12: Using the method of successive approximation, solve the following integral equation:

$$g(x) = 1 + \int_0^x (x-t).g(t)dt$$

Taking $g_0(x) = 0$.

Solution: The given equation is

$$g(x) = 1 + \int_0^x (x-t).g(t)dt \tag{i}$$

Comparing Eq. (i) with Eq. (6.101) [$y(x)$ is $g(x)$],

$$f(x) = 1,\ \lambda = 1, K(x,t) = (x-t)$$

The nth order approximation,

$$g_n(x) = f(x) + \lambda \int_0^x K(x,t).g_{n-1}(t).dt$$

$$g_n(x) = 1 + \int_0^x (x-t).g_{n-1}(t).dt \tag{ii}$$

Let $n = 1$, $$g_1(x) = 1 + \int_0^x (x-t).0\ dt = 1$$

Next, let $n = 2$ in Eq. (ii); using $g_1(x) = 1$,

$$g_2(x) = 1 + \int_0^x (x-t).1\ dt = 1 + x^2 - \frac{x^2}{2} = 1 + \frac{x^2}{2!},$$

Then, putting $n = 3$ in Eq. (ii) and using $g_2(x)$, we get

$$g_3(x) = 1 + \int_0^x (x-t)\left(1 + \frac{x^2}{2!}\right) dt = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

In general, we have

$$g_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n-2}}{(2n-2)!}$$

Note: As $n \to \infty$, we get the infinte term to get $\cosh x$.
Hence, the required sum is $\lim_{n\to\infty} g_n(x) = \cosh x$.

EXAMPLE 6.13: Using the method of successive approximation, solve the following integral equation.

$$g(x) = x - \int_0^x (x-t)g(t)\,dt, \quad g_0(t) = 0$$

Solution: The given equations are

$$g(x) = x - \int_0^x (x-t)g(t)\,dt \tag{i}$$

and $$g_0(t) = 0 \tag{ii}$$

Comparing Eqs. (i), (ii) with Eq. (6.101),

[here, $y(x) = g(x)$], $f(x) = x, \lambda = -1, K(x,t) = x - t, g_0(x) = 0$

The nth order approximation is given by

$$g_n(x) = f(x) + \lambda \int_0^x K(x,t).\ g_{n-1}(t)\,dt$$

or $$g_n(x) = x + (-1)\int_0^x (x-t).\ g_{n-1}(t)\,dt \tag{iii}$$

Now, for $g_1(x)$, we put $n = 1$ in Eq. (iii) and take $g_0(x) = 0$.

We get $$g_1(x) = x \tag{iv}$$

For $g_2(x)$, we take $n = 2$, and use $g_1(x) = x$, and obtain

$$g_2(x) = x - \int_0^x (x-t).\ t\ dt = x - \frac{x^3}{2} + \frac{x^3}{3} = x - \frac{x^3}{3!}$$

Similarly, for $g_3(x)$, we take $n = 3$ in Eq. (iii) and using $g_2(x)$, we get

$$g_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

In general, $$g_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

As the required solution is $\lim\limits_{n\to\infty} g_n(x)$, we get

$$\lim_{n\to\infty} g_n(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} + \cdots \infty = \sin x$$

EXAMPLE 6.14: Using the method of successive approximation, solve the integral equation

$$g(x) = \int_0^x \frac{1+g^2(t)}{1+t^2}\,dt$$

taking the zero order approximation (a), $g_0(x) = 0$, and (b) $g_0(x) = x$.

Solution: (a) The given equations is

$$g(x) = \int_0^x \frac{1+g^2(t)}{1+t^2}\,dt \qquad \text{(i)}$$

Equation (i) is non-linear.
Comparing it with Eq. (6.101), we have

$$f(x) = 0, \lambda = 1, K(x,t).\ g(t) = \frac{1+g^2(t)}{1+t^2}$$

and let $$g_n(x) = \int_0^x K(x,t).g_{n-1}(t)dt$$

$$g_n(x) = \int_0^x \frac{1+g_{n-1}^2(t)}{1+t^2}\,dt \qquad \text{(ii)}$$

so that $$g_1(x) = \int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1} x$$

$$g_2(x) = \int_0^x \frac{1+(\tan^{-1} t)^2}{1+t^2}\,dt = \tan^{-1} x + \frac{1}{3}(\tan^{-1} x)^3$$

$$g_3(x) = \int_0^x \frac{1+\left\{\tan^{-1} t + \frac{1}{3}(\tan^{-1} t)^3\right\}^2}{1+t^2}\,dt$$

$$g_3(x) = \tan^{-1} x + \frac{1}{3}(\tan^{-1} x)^3 + \frac{2}{3\times 5}(\tan^{-1} x)^5 + \frac{1}{7\times 9}(\tan^{-1} t)^7$$

Now, $$g(x) = \lim_{n\to\infty} g_n(x) = \tan(\tan^{-1} x) = x$$

(In the expansion of $\tan x_1$, if we put $x_1 = \tan^{-1}x$, we get it)

(b) If we take $g_0(x) = x, g_1(x) = x = g_2(x) = g_3(x) = \cdots = g_n(x)$

So, $g(x) = \lim_{n\to\infty} (x) = x$

EXERCISE 6.3

1. Find the resolvent kernel, given (a) $K(x,t) = 1$, (b) $K(x,t) = e^{x-t}$.
2. With the help of resolvent kernel, solve

   (a) $y(x) = f(x) + \lambda\int_0^x e^{x-t} y(t)dt$

   (b) $y(x) = \sin x + 2\int_0^x e^{x-t} y(t)dt$

   (c) $y(x) = 1 + \int_0^x y(t)dt$,

   (d) $y(x) = x + \int_0^x (t-x).y(t)dt$

   (e) $y(x) = 1 + \int_0^x (t-x).y(t)dt$

   (f) $g(x) = \cos x - x - 2 + \int_0^x (t-x).g(t)dt$
3. Using the method of successive approximation, find the solution of

   (a) $y(x) == 1 + \int_0^x y(t)\,dt, y_0(x) = 0$ [RU 94, 03]

   (b) $y(x) = 1 + x - \int_0^x y(t)\,dt, y_0(x) = 1$

   (c) $y(x) = x.2^x - \int_0^x 2^{x-t}.g(t)dt, g_0(x) = x.2^x$

   Recollect the following expansions:

   (a) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = \lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$

   (b) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + (-1)^{n-1}\dfrac{x^n}{n} + \cdots$

   (c) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + \sin\left(\dfrac{n\pi}{2}\right)\dfrac{.x^n}{n!} + \cdots, = \sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k+1)!}x^{2k+1}$

   (d) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots + \cos\left(\dfrac{n\pi}{2}\right)\dfrac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!}$

   (e) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots$ for $|x| < \dfrac{\pi}{2}$

   (f) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!}$ for all x

(g) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$ for all x

(h) $\tanh x = x - \dfrac{x^3}{3!} + \dfrac{2x^5}{15} - \dfrac{17x^7}{315} + \cdots$ for $|x| < \dfrac{\pi}{2}$

(i) $\sin^{-1} x = x - \dfrac{1}{2}.\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\dfrac{x^5}{5} - \dfrac{1.3.5}{2.4.6}.\dfrac{x^7}{7} + \cdots (-1 \le x \le 1)$

(j) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots (-1 < x < 1)$

Answers:

1. (a) $\exp.\{\lambda(x-t)\}$, (b) $\exp.\{(x-t)(1+\lambda)\}$

2. (a) $y(x) = f(x) + \lambda \int_0^x e^{(x-t)(1+\lambda)} f(t)dt$

   (b) $y(x) = \dfrac{1}{5}e^{3x} + \dfrac{2}{5}\sin x - \dfrac{1}{5}\cos x$

   (c) $y = e^x$

   (d) $y(x) = \sin x$

   (e) $y(x) = \cos x$

   (f) $g(x) = -\cos x - \sin x - \dfrac{1}{2}x \sin x$

3. (a) $y(x) = e^x$

   (b) $y(x) = 1$

   (c) $g(x) = 2^x(1-e^{-x})$

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CHAPTER 7

Classical Fredholm Theory

7.1 INTRODUCTION

The solution of the Fredholm integral equation of the second kind, i.e.,

$$g(x)=f(x)+\lambda\int_a^b K(x,t)\,g(t)\,dt \tag{7.1}$$

has been discussed in Chapter 6 as a uniformly convergent power series in parameter λ for suitably small value of $|\lambda|$. As a matter of fact, Fredholm obtained the solution of Eq. (7.1) in general form, which is valid for all values of parameter λ. These solutions are contained in three theorems, which are known as *Fredholm's first, second and third fundamental theorems*.

In this chapter, we shall study Eq. (7.1) when the functions $f(x)$ and kernel $K(x,t)$ are any integrable functions. Moreover, the present method enables us to get explicit formulae for the solution in terms of certain determinants.

7.2 FREDHOLM'S FIRST THEOREM

Statement: The non-homogeneous Fredholm integral equation of the second kind

$$g(x)=f(x)+\lambda\int_a^b K(x,t)\,g(t)\,dt \tag{7.2}$$

where the functions $f(x)$ and $K(x,t)$ are integrable, has a unique solution

$$g(x)=f(x)+\lambda\int_a^b R(x,t;\lambda)\,f(t)\,dt \tag{7.3}$$

where the resolvent kernel $R(x,t;\lambda)$ is a meromorphic function of parameter λ defined by

$$R(x,t;\lambda)=\frac{D(x,t;\lambda)}{D(\lambda)}; \qquad [D(\lambda)\neq 0] \tag{7.4}$$

$D(x,t;\lambda)$ and $D(\lambda)$ are entire* functions of parameter λ defined by Fredholm's series of the form

$$D(x,t;\lambda)=K(x,t)+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}\int\cdots\int K\begin{pmatrix} z,z_1\ldots,z_m \\ t,z_1\ldots,z_m \end{pmatrix}dz_1\cdots dz_m \tag{7.5}$$

and
$$D(\lambda)=1+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}\int\cdots\int K\begin{pmatrix} z_1,\ldots,z_m \\ z_1,\ldots,z_m \end{pmatrix}dz_1\cdots dz_m \tag{7.6}$$

both of which converge for all values of λ. In particular, the solution of homogeneous integral equation is identically zero.

Moreover, it is to be noted that

$$\begin{vmatrix} K(x_1,t_1) & K(x_1,t_2) & \cdots & K(x_1,t_n) \\ K(x_2,t_1) & K(x_2,t_2) & \cdots & K(x_2,t_n) \\ : & : & : & : \\ K(x_n,t_1) & K(x_n,t_2) & \cdots & K(x_n,t_n) \end{vmatrix}=K\begin{pmatrix} x_1,x_2,\ldots,x_n \\ t_1,t_2,\ldots,t_n \end{pmatrix} \tag{7.7}$$

is known as the **Fredholm determinant**.

Proof: Let the interval (a, b) be partitioned into n equal parts by the points

$$x_1=t_1=a, x_2=t_2=a+h,\ldots,x_n=t_n=a+(n-1)h \tag{7.8}$$

where,

$$h=\frac{(b-a)}{n}$$

In this way, the approximation formula is developed as

$$\int_a^b K(x,t)\,g(t)\,dt \simeq h\sum_{j=1}^{n}K(x,x_j)\,g(x_j) \tag{7.9}$$

Hence, Eq. (7.2) reduces to

$$g(x)=f(x)+\lambda h\sum_{j=1}^{n}K(x,x_j)\,g(x_j) \tag{7.10}$$

Since Eq. (7.10) holds for all values of x in the internal (a, b), in particular, it must be satisfied at the n points of division $x_i (i = 1, 2, \ldots, n)$, and thus, the following system of equations is obtained:

$$g(x_i)=f(x_i)+\lambda h\sum_{j=1}^{n}K(x_i,x_j)\,g(x_j),\ \ (i=1,2,\ldots,n) \tag{7.11}$$

Now, we express $\quad g(x_i)=g_i, f(x_i)=f_i, K(x_i,x_j)=K_{ij} \qquad (7.12)$

* Refer to author's Complex Variable by Ramesh Book Depot.

Then, Eq. (7.11) gives an approximation for Eq. (7.2) in terms of the system of n linear equations with n unknowns $g_1, g_2, \ldots, g_n$ as

$$g_i - \lambda h \sum_{j=1}^{n} K_{ij} g_i = f_i \qquad (i = 1, 2, \ldots, n) \tag{7.13}$$

And Eq. (7.13) can be rewritten as

$$\left.\begin{array}{lll}
(1-\lambda hK_{11})g_1 - \lambda hK_{12}g_2 - & \cdots & -\lambda hK_{1n}g_n = f_1 \\
-\lambda hK_{21}g_1 + (1-\lambda hK_{22})g_2 - & \cdots & -\lambda hK_{2n}g_n = f_2 \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
-\lambda hK_{n1}g_1 - \lambda hK_{n2}g_2 - & \cdots & +(1-\lambda hK_{nn})g_n = f_n
\end{array}\right\} \tag{7.14}$$

The values of $g_1, g_2, \ldots, g_n$ obtained by solving the algebraic system Eq. (7.14) are approximate solutions of Eq. (7.2) at the points $x_1, x_2, \ldots, x_n$. These solutions $g_1, g_2, \ldots, g_n$ can be plotted as ordinates, and by interpolation, we can draw a curve $g(x)$ which is an approximation to the actual solution.

The solutions $g_1, g_2, \ldots, g_n$ obtained by solving the algebraic system of equations [i.e., Eq. (7.14)] may be expressed in the form of the ratios of certain determinants, with the resolvent determinant $D_n(\lambda)$ of the above algebraic system [i.e., Eq. (7.14)], where

$$D_n(\lambda) = \begin{vmatrix}
1-\lambda hK_{11} & -\lambda hK_{12} & \cdots & -\lambda hK_{1n} \\
-\lambda hK_{21} & 1-\lambda hK_{22} & \cdots & -\lambda hK_{2n} \\
\vdots & \vdots & \vdots & \vdots \\
-\lambda hK_{n1} & -\lambda hK_{n2} & \cdots & 1-\lambda hK_{nn}
\end{vmatrix} \tag{7.15}$$

provided that $D_n(\lambda) \neq 0$.
The approximate eigenvalues are obtained by setting this determinant to zero. Now, expanding $D_n(\lambda)$ in powers of the quantity $(-\lambda h)$, it is found that the first term not containing this factor is equal to unity. The term containing $(-\lambda h)$ in the first power is the sum of all the determinants containing only one column $-\lambda hK_{rs}$, $r = 1, \ldots, n$. Considering the contribution from all these columns $s = 1, \ldots, n$, we find that the total contribution is $-\lambda h \sum_{s=1}^{n} K_{ss}$.

The term containing the factor $(-\lambda h)^2$ is the sum of all the determinants having two columns containing that factor. This gives rise to the determinants of the form

$$(-\lambda h)^2 \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

where (p, q) is an arbitrary pair of integers chosen from the sequence $1, \ldots, n$ with $p < q$.

Similarly, the term containing the factor $(-\lambda h)^3$ is the sum of the determinants of the form

$$(-\lambda h)^3 \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix}$$

where (p, q, r) is an arbitrary triplet of integers taken from the sequence $1, 2, ..., n$ with $p < q < r$.

Proceeding likewise, we obtain the remaining terms in the expansion of $D_n(\lambda)$. Thus, the determinant [Eq. (7.15)] may be expressed in the following form:

$$D_n(\lambda) = 1 - \lambda h \sum_{s=1}^{n} K_{ss} + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

$$+ \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix} + \cdots$$

$$+ \frac{(-\lambda h)^n}{n!} \sum_{p_1, p_2, ..., p_n = 1}^{n} \begin{vmatrix} K_{p_1 p_1} & K_{p_1 p_2} & \cdots & K_{p_1 p_n} \\ K_{p_2 p_1} & K_{p_2 p_2} & \cdots & K_{p_2 p_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K_{p_n p_1} & K_{p_n p_2} & \cdots & K_{p_n p_n} \end{vmatrix}$$

$$D_n(\lambda) = 1 - \lambda h \sum_{m=1}^{n} K(x_p, x_p) + \frac{(-\lambda h)^2}{2\,!} \sum_{p,q=1}^{n} K\begin{pmatrix} x_p & x_q \\ x_p & x_q \end{pmatrix}$$

$$+ \frac{(-\lambda h)^3}{3\,!} \sum_{p,q,r=1}^{n} K\begin{pmatrix} x_p, x_q, x_r \\ x_p, x_q, x_r \end{pmatrix} + \cdots \tag{7.16}$$

[By applying Eq. (7.7)]

Since $\lim\limits_{n\to\infty} h = \lim\limits_{n\to\infty} \dfrac{(b-a)}{n} = 0$ and each term of the sum shown in Eq. (7.16) tends to be single, double, triple integral, etc., therefore we have

$$D(\lambda) = 1 - \lambda \int_a^b K(x, x)\, dx + \frac{\lambda^2}{2\,!} \int_a^b \int_a^b K\begin{pmatrix} x_1, & x_2 \\ x_1, & x_2 \end{pmatrix} dx_1\, dx_2$$

$$- \frac{\lambda^3}{3\,!} \int_a^b \int_a^b \int_a^b K\begin{pmatrix} x_1, & x_2, & x_3 \\ x_1, & x_2, & x_3 \end{pmatrix} dx_1\, dx_2\, dx_3 + \cdots \tag{7.17}$$

where $D(\lambda)$ is called Fredholm's determinant and the series occurring on the R.H.S. of Eq. (7.17) is called **Fredhlom's first series.**

It may be pointed out that Hilbert gave a rigorous proof of the fact that sequence $D_n(\lambda) \to D(\lambda)$ in the limit and Fredholm proved the convergence of Eq. (7.17) for all values of λ by using the fact that kernel $K(x,t)$ is bounded and is an integrable function. Thus, $D(\lambda)$ is an entire function of the complex parameter λ.

Now, we can find the solution of Eq. (7.2) in the form given by Eq. (7.3), where the resolvent kernel is the quotient of $D(x,t;\lambda)$ and $D(\lambda)$. For this, we determine $D(x,t;\lambda)$ as the sum of certain functional series. It is known that the resolvent kernel $R(x,t;\lambda)$ satisfies the following relation:

$$R(x,t;\lambda)=K(x,t)+\lambda\int_a^b K(x,z)\,R(z,t;\lambda)\,dz \tag{7.18}$$

From Eqs. (7.6) and (7.18), it follows that

$$\frac{D(x,t;\lambda)}{D(\lambda)}=K(x,t)+\lambda\int_a^b K(x,z)\frac{D(z,t;\lambda)}{D(\lambda)}\,dz, \qquad \{D(\lambda)\neq 0\}$$

or

$$D(x,t;\lambda)=K(x,t)\,D(\lambda)+\lambda\int_a^b K(x,z)\,D(z,t;\lambda)\,dz \tag{7.19}$$

Now, the form of the series shown in Eq. (7.17) for $D(\lambda)$ suggests that we search the solution of Eq. (7.19) in the form of a power series in parameter λ, i.e.,

$$D(x,t;\lambda)=B_0(x,t)+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m\,!}B_m(x,t) \tag{7.20}$$

For this, rewriting Eq. (7.17) as

$$D(\lambda)=1+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m\,!}\mu_m \tag{7.21}$$

where,

$$\mu_m=\int_a^b\cdots\int_a^b K\begin{pmatrix}x_1,x_2,\ldots,x_m\\ x_1,x_2,\ldots,x_m\end{pmatrix}dx_1\cdots dx_m \tag{7.22}$$

Now, substituting the series for $D(x,t;\lambda)$ and $D(\lambda)$ from Eqs. (7.20) and (7.21) in Eq. (7.19) and comparing the coefficients of equal powers of λ, we derive the following recursion relations:

$$B_0\,(x,t)=K(x,t) \tag{7.23}$$

and

$$B_m(x,t)=\mu_m K(x,t)-m\int_a^b K(x,z)\,B_{m-1}(z,t)\,dz \tag{7.24}$$

Now, we shall show that for each $m(m = 1,2,3,....)$,

$$B_m(x,t)=\int_a^b\cdots\int_a^b K\begin{pmatrix} x,z_1,z_2,\dots,z_m \\ t,z_1,z_2,\dots,z_m \end{pmatrix} dz_1\dots dz_m \tag{7.25}$$

First, we see that for $m = 1$, Eq. (7.24) takes the following form:

$$B_1(x,t)=\mu_1 K(x,t)-\int_a^b K(x,z)B_0(z,t)\,dz$$

or

$$B_1(x,t)=K(x,t)\int_a^b K(z,z)\,dz-\int_a^b K(x,z)K(z,t)\,dz$$

or

$$B_1(x,t)=\int_a^b K\begin{pmatrix} x,z \\ t,z \end{pmatrix} dz \tag{7.26}$$

It shows that Eq. (7.25) holds for $m = 1$.

To prove that Eq. (7.25) holds for general m, we expand the determinant under the integral sign by the relation

$$K\begin{pmatrix} x_1,z_1,z_2,\dots,z_m \\ t,z_1,z_2,\dots,z_m \end{pmatrix}=\begin{vmatrix} K(x,t) & K(x,z_1) & \cdots & K(x,z_m) \\ K(z_1,t) & K(z_1,z_1) & \cdots & K(z_1,z_m) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_m,t) & K(z_m,z_1) & \cdots & K(z_m,z_m) \end{vmatrix} \tag{7.27}$$

with respect to the elements of the given row, transposing in turn the first column one place to the right, integrating both sides and using Eq. (7.22), the result, i.e., Eq. (7.25) follows by mathematical induction.

Now, from Eqs. (7.21), (7.23) and (7.25), we derive the so–called **Fredholm's second series**.

$$D(x,t;\lambda)=K(x,t)+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m\,!}\int_a^b\dots\int_a^b K\begin{pmatrix} x,z_1,\dots,z_m \\ t,z_1,\dots,z_m \end{pmatrix} dz_1\dots dz_m \tag{7.28}$$

Equation (7.28) converges for all values of λ.

In the end, it will be shown that the solution in the form obtained by Fredholm is unique and is given by Eq. (7.3). Before doing this, we find that the integral equation [i.e., Eq. (7.18)] satisfied by $R(x,t;\lambda)$ is valid for all values of λ for which $D(\lambda) \neq 0$. Through Chapter 6, we already know that Eq. (7.18) holds for $(\lambda) < B^{-1}$, where

$$B=\left[\int_a^b\int_a^b |K(x,t)|^2\,dx\,dt\right]^{1/2}$$

Since both sides of Eq. (7.18) are thus found to be meromorphic, the result follows.

To establish the uniqueness of the solution of Eq. (7.2), it is assumed that $g(x)$ is a solution of Eq. (7.2), provided that $D(\lambda) \neq 0$. Rewriting (7.2) as

$$g(z) = f(z) + \lambda \int_a^b K(z,t)\, g(t)\, dt \tag{7.29}$$

Multiplying both sides of Eq. (7.29) by $R(x, z; \lambda)$, and then, integrating both sides with respect to 'z' from a to b, we get

$$\int_a^b R(x,z;\lambda)\, g(z)\, dz = \int_a^b R(x,z;\lambda)\, f(z)\, dz + \lambda \int_a^b \left[\int_a^b R(x,z;\lambda) K(z,t)\, dz \right] g(t)\, dt \tag{7.30}$$

Using Eq. (7.18), we have

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b R(x,z;\lambda)\, K(z,t)\, dz$$

or
$$\lambda \int_a^b R(x,z;\lambda) K(z,t)\, dz = R(x,t;\lambda) - K(x,t) \tag{7.31}$$

From Eqs. (7.30) and (7.31), we have

$$\int_a^b R(x,z;\lambda)\, g(z) dz = \int_a^b R(x,z;\lambda)\, f(z)\, dz + \int_a^b [R(x,t;\lambda) - K(x,t)]\, g(t)\, dt$$

or
$$\int_a^b R(x,t;\lambda)\, g(t) dt = \int_a^b R(x,t;\lambda)\, f(t)\, dt + \int_a^b R(x,t;\lambda)\, g(t)\, dt - \int_a^b K(x,t)\, g(t)\, dt$$

or
$$\int_a^b K(x,t)\, g(t)\, dt = \int_a^b R(x,t;\lambda)\, f(t)\, dt \tag{7.32}$$

From Eq. (7.2), we have

$$\int_a^b K(x,t)\, g(t)\, dt = \frac{g(x) - f(x)}{\lambda} \tag{7.33}$$

and from Eqs. (7.32) and (7.33), we have

$$\frac{g(x) - f(x)}{\lambda} = \int_a^b R(x,t;\lambda)\, f(t)\, dt$$

or
$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt \tag{7.34}$$

and this form is unique.

7.3 WORKING RULE FOR EVALUATING THE RESOLVENT KERNEL AND SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BY USING FREDHOLM'S FIRST THEOREM

Let the Fredholm integral equation be

$$g(x)=f(x)+\lambda\int_a^b K(x,t)\,g(t)\,dt \tag{7.35}$$

The Fredholm's first theorem enables us to get explicit formulae for the solution of Eq. (7.35) in terms of determinants.

We know that the unique and continuous solution of Eq. (7.35) is

$$g(x)=f(x)+\lambda\int_a^b R(x,t;\lambda)\,f(t)\,dt \tag{7.36}$$

where the function $R(x,t;\lambda)$ is called Fredholm resolvent kernel of Eq. (7.35) and is defined by

$$\frac{D(x,t;\lambda)}{D(\lambda)} \tag{7.37}$$

provided that $D(\lambda)\neq 0$.

Here,
$$D(x,t;\lambda)=K(x,t)+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m\,!}B_m(x,t) \tag{7.38}$$

and
$$D(\lambda)=1+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m\,!}\mu_m \tag{7.39}$$

wherein the coefficients are given by

$$B_n(x,t)=\int_a^b\cdots\int_a^b \begin{vmatrix} K(x,t) & K(x,z_1) & \cdots & K(x,z_n) \\ K(z_1,t) & K(z_1,z_1) & \cdots & K(z_1,z_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,t) & K(z_n,z_1) & \cdots & K(z_n,z_n) \end{vmatrix} dz_1\ldots dz_n \tag{7.40}$$

and
$$B_0(x,t) = K(x,t) \tag{7.41}$$

and
$$\mu_n=\int_a^b\cdots\int_a^b \begin{vmatrix} K(z_1,z_1) & K(z_1,z_2) & \cdots & K(z_1,z_n) \\ K(z_2,z_1) & K(z_2,z_2) & \cdots & K(z_2,z_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(z_n,z_1) & K(z_n,z_2) & \cdots & K(z_n,z_n) \end{vmatrix} dz_1\ldots dz_n \tag{7.42}$$

The function $D(x, t; \lambda)$ is called **Fredholm minor** and $D(\lambda)$ is the **Fredholm determinant**.

In case, kernel $K(x, t)$ is bounded or the integral

$$\int_a^b \int_a^b K^2(x,t)\,dx\,dt$$

has a finite value, Eqs. (7.38) and (7.39) converge for all values of λ, and therefore, are entire functions for λ.

Also, the resolvent kernel $R(x, t; \lambda)$ is an analytic function of λ, except for those values of λ which are the zeros of the function $D(\lambda)$. Obviously, the latter are poles of the resolvent kernel $R(x, t; \lambda)$.

Alternative method for calculating $B_m(x,t)$ and μ_n

The coefficients μ_n and the function $B_m(x, t)$ are also found from the following recurrence relations:

We have

$$\mu_0 = 1, \mu_n = \int_a^b B_{n-1}(s,s)\,ds, \tag{7.43}$$

and

$$B_n(x,t) = \mu_n K(x,t) - \int_a^b K(x,z)\, B_{n-1}(z,t)\,dz, \quad m \ge 1 \tag{7.44}$$

Since $\mu_0 = 1$ and $B_0(x, t)$, is directly known, therefore we can use Eqs. (7.43) and (7.44) to find, in succession $\mu_1, B_1(x,t); \mu_2, B_2(x,t)$, and so on. Continuing in this way, all the coefficients can be calculated. In certain cases, depending on the explicit form of the kernel, Eqs. (7.38) and (7.39) contain only a finite number of terms.

It is to be kept in mind that one distinct advantage of Fredholm method is that Eq. (7.37) is uniformly convergent for all values of λ unless $D(\lambda) = 0$.

EXAMPLE 7.1: Find the resolvent kernel of the following kernels by using Fredholm determinants:

(a) $K(x, t) = xe^t$; $a = 0$, $b = 1$
(b) $K(x, t) = 2x - t$; $0 \le x \le 1$, $0 \le t \le 1$,
(c) $K(x, t) = \sin x \cos t$; $0 \le x \le 2\pi$, $0 \le t \le 2\pi$,

Solution: (a) Here, $K(x, t) = xe^t$

Now, by Eq. (7.24), $B_0(x, t) = K(x, t) = xe^t$

Also, by Eq. (7.26) $$B_1(x,t) = \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z_1,t) & K(z_1,z_1) \end{vmatrix} dz_1$$

$$B_1(x,t) = \int_0^1 \begin{vmatrix} xe^t & xe^{z_1} \\ z_1 e^t & z_1 e^{z_1} \end{vmatrix} dz_1 \qquad \text{[Using Eq. (7.40)]}$$

$$B_1(x, t) = 0$$

(since the columns of the determinat under the integral sign are identical)

Similarly, $$B_2(x,t) = \int_0^1\int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) & K(x,z_2) \\ K(z_1,t) & K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,t) & K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1 dz_2$$

$$B_2(x,t) = \int_0^1\int_0^1 \begin{vmatrix} xe^t & xe^{z_1} & xe^{z_2} \\ z_1e^t & z_1e^{z_1} & z_1e^{z_2} \\ z_2e^t & z_2e^{z_1} & z_2e^{z_2} \end{vmatrix} dz_1 dz_2 = 0$$

Since $B_1(x,t) = B_2(x,t) = 0$, it follows that $B_n(x,t) = 0$, for $n \le 1$. Thus, from Eq. (7.42), we have

$$\mu_1 = \int_0^1 K(z_1,z_1)dz_1 = \int_0^1 z_1 e^{z_1} dz_1$$

$$\mu_1 = [z_1 e^{z_1}]_0^1 - \int_0^1 e^{z_1} dz_1$$

$$\mu_1 = e - [e^{z_1}]_0^1 = e - (e-1) = 1$$

and $$\mu_2 = \int_0^1\int_0^1 \begin{vmatrix} z_1e^{z_1} & z_1e^{z_2} \\ z_2e^{z_1} & z_2e^{z_2} \end{vmatrix} dz_1 dz_2 = 0$$

Clearly, $\mu_m = 0$ for all $m \ge 2$.

Now, by Eq. (7.5), $D(x,t;\lambda), = K(x,t) - \lambda B_1(x,t) + \frac{\lambda^2}{2!} B_2(x,t) - \cdots$

$D(x,t;\lambda) = xe^t$ (By substituting values of $K(x,t)$ $B_1(x,t)$, etc.)

and by Eq. (7.6), $D(\lambda) = 1 - \lambda\mu_1 + \frac{\lambda^2}{2!}\mu_2 - \cdots = 1 - \lambda$

(By substituting the values of μ_1, μ_2, etc.)

Thus, the Fredholm resolvent kernel as given by Eq. (7.4) is

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \frac{xe^t}{1-\lambda}$$

(b) Here, we have $B_0(x,t) = K(x,t) = 2x - t$; and then, similar to part (a),

$$B_1(x,t) = \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z_1,t) & K(z_1,z_1) \end{vmatrix} dz_1 = \int_0^1 \begin{vmatrix} 2x-t & 2x-z_1 \\ 2z_1 - t & 2z_1 - z_1 \end{vmatrix} dz_1$$

$$B_1(x,t) = \int_0^1 [z_1(2x-t) - (2x-z_1)(2z_1-t)]dz_1$$

or $$B_1(x,t)=\int_0^1[z_1(2x-t)-(4xz_1-2xt-2z_1^2+z_1t)]dz_1$$

or $$B_1(x,t)=\int_0^1[2z_1^2+z_1(2x-t-4x-t)+2xt]dz_1$$

or $$B_1(x,t)=\left[\frac{2z_1^3}{3}+\frac{z_1^2}{2}(-2x-2t)+2xtz_1\right]_0^1=\frac{2}{3}-x-t+2xt$$

Also, $$B_2(x,t)=\int_0^1\int_0^1\begin{vmatrix}K(x,t) & K(x,z_1) & K(x,z_2)\\ K(z_1,t) & K(z_1,z_1) & K(z_1,z_2)\\ K(z_2,t) & K(z_2,z_1) & K(z_2,z_2)\end{vmatrix}dz_1dz_2$$

$$B_2(x,t)=\int_0^1\int_0^1\begin{vmatrix}2x-t & 2x-z_1 & 2x-z_2\\ 2z_1-t & 2z_1-z_1 & 2z_1-z_2\\ 2z_2-t & 2z_2-z_1 & 2z_2-z_2\end{vmatrix}dz_1dz_2$$

or $B_2(x,t) = 0.$ (After little simplification)

Hence, $B_p(x,t) = 0$ for all $p \geq 2$.

Again, $$\mu_1=\int_0^1K(z_1,z_1)\,dz_1=\int_0^1(2z_1-z_1)dz_1=\left[\frac{1}{2}z_1^2\right]_0^1=\frac{1}{2}$$

and $$\mu_2=\int_0^1\int_0^1\begin{vmatrix}K(z_1,z_1) & K(z_1,z_2)\\ K(z_2,z_1) & K(z_2,z_2)\end{vmatrix}dz_1dz_2$$

$$\mu_2=\int_0^1\int_0^1\begin{vmatrix}2z_1-z_1 & 2z_1-z_2\\ 2z_2-z_1 & 2z_2-z_2\end{vmatrix}dz_1dz_2=\frac{1}{3}\text{ (Upon simplification)}$$

and $$\mu_3=\int_0^1\int_0^1\int_0^1\begin{vmatrix}K(z_1,z_1) & K(z_1,z_2) & K(z_1,z_3)\\ K(z_2,z_1) & K(z_2,z_2) & K(z_2,z_3)\\ K(z_3,z_1) & K(z_3,z_2) & K(z_3,z_3)\end{vmatrix}dz_1dz_2dz_3$$

$$\mu_3=\int_0^1\int_0^1\int_0^1\begin{vmatrix}2z_1-z_1 & 2z_1-z_2 & 2z_1-z_3\\ 2z_2-z_1 & 2z_2-z_2 & 2z_2-z_3\\ 2z_3-z_1 & 2z_3-z_2 & 2z_3-z_3\end{vmatrix}dz_1dz_2dz_3$$

$\mu_3 = 0$ (Upon simplification).

Hence, $\mu_p = 0$ for all $p \geq 3$.

Thus, we have upon using Eq. (7.5),

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) = K(x,t) - \lambda B_1(x,t)$$

$$D(x,t;\lambda) = 2x - t - \lambda\left(\frac{2}{3} - x - t + 2xt\right)$$

and by using Eq. (7.6),

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!}\mu_m = 1 - \lambda\mu + \frac{\lambda^2}{2!}\mu_2 = 1 - \frac{1}{2}\lambda + \frac{\lambda^2}{6}$$

The Fredholm resolvent kernel is given by Eq. (7.4) providing

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \frac{2x - t - \lambda\left(\frac{2}{3} - x - t + 2xt\right)}{1 - (\lambda/2) + (\lambda^2/6)}$$

(c) Here, $B_0(x,t) = K(x,t) = \sin x \cos t$. As before

$$B_1(x,t) = \int_0^{2\pi} \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z_1,t) & K(z_1,z_1) \end{vmatrix} dz_1$$

$$B_1(x,t) = \int_0^{2\pi} \begin{vmatrix} \sin x \cos t & \sin x \cos z_1 \\ \sin z_1 \cos t & \sin z_1 \cos z_1 \end{vmatrix} dz_1 = 0$$

Hence, $B_p(x,t) = 0$ for all $p \geq 2$.

Next, $$\mu_1 = \int_0^{2\pi} K(z_1,z_1)dz_1 = \int_0^{2\pi} \sin z_1 \cos z_1 \; dz_1 = 0$$

Hence, $\mu_p = 0$ for all $p \geq 2$.
Thus, by using Eq. (7.5), we have

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t)$$

$$D(x,t;\lambda) = K(x,t) - \lambda B_1(x,t) + \cdots$$

$D(x,t;\lambda) = \sin x \cos t$ (By substituting the above values)

By using Eq. (7.6),

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!}\mu_m$$

$$D(\lambda) = 1 - \lambda\mu_1 + \cdots = 1$$

(Upon substituting the above values)

The Fredholm resolvent kernel is given by Eq. (7.4), providing

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \sin x \ \cos t$$

EXAMPLE 7.2: Find the resolvent kernel of the following kernels by using the Fredholm determinants:

(a) $K(x,t) = 1 + 3xt, \quad 0 \le x \le 1, \ 0 \le t \le 1$

(b) $K(x,t) = x^2t - xt^2, \quad 0 \le x \le 1, \ 0 \le t \le 1$

Solution: (a) Here, $B_0(x,t) = K(x,t) = 1 + 3xt$. Now,

$$B_1(x,t) = \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z_1,t) & K(z_1,z_1) \end{vmatrix} dz_1$$

$$B_1(x,t) = \int_0^1 \begin{vmatrix} 1+3xt & 1+3xz_1 \\ 1+3z_1t & 1+3z_1^2 \end{vmatrix} dz_1$$

$$B_1(x,t) = -\left[\frac{3(x+t)}{2} - 3xt - 1\right] \text{ (Upon simplification)}$$

Also, $$B_2(x,t) = \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) & K(x,z_2) \\ K(z_1,t) & K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,t) & K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1\, dz_2$$

$$B_2(x,t) = \int_0^1\int_0^1 \begin{vmatrix} 1+3xt & 1+3xz_1 & 1+3xz_2 \\ 1+3z_1t & 1+3z_1^2 & 1+3z_1z_2 \\ 1+3z_2t & 1+3z_2z_1 & 1+3z_2^2 \end{vmatrix} dz_1\, dz_2$$

$B_2(x,t) = 0$ (Upon simplification)

Hence, $B_p(x,t) = 0$ for all $p \ge 2$.

Next, $$\mu_1 = \int_0^1 K(z_1,z_1)dz_1 = \int_0^1 (1+3z_1^2)\, dz_1 = 2$$

$$\mu_2 = \int_0^1\int_0^1 \begin{vmatrix} K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1 dz_2$$

$$\mu_2 = \int_0^1\int_0^1 \begin{vmatrix} 1+3z_1^2 & 1+3z_1z_2 \\ 1+3z_2z_1 & 1+3z_2^2 \end{vmatrix} dz_1 dz_2$$

$$\mu_2 = \frac{1}{2}, \text{ (upon simplification)}$$

Also, $\mu_3 = 0$ (can be verified by readers)

Hence, $\mu_p = 0$ for all $p \ge 3$.

Now, we have $$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t)$$

$$D(x,t;\lambda) = 1 + 3xt + \lambda\left[\frac{3(x+t)}{2} - 3xt - 1\right]$$

and $$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m$$

$$D(\lambda) = 1 - \lambda\mu_1 + \frac{\lambda^2}{2!}\mu_2 - \cdots = 1 - 2\lambda + \frac{\lambda^2}{4}$$

The Fredholm resolvent kernel is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}$$

$$R(x,t;\lambda) = \frac{1 + 3xt + \lambda\left[\frac{3(x+t)}{2} - 3xt - 1\right]}{1 - 2\lambda - (1/4)\lambda^2}$$

(b) Here, $B_0(x,t) = K(x,t) = x^2t - xt^2$. Then, as before

$$B_1(x,t) = \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z_1,t) & K(z_1,z_1) \end{vmatrix} dz_1$$

$$B_1(x,t) = \int_0^1 \begin{vmatrix} x^2t - xt^2 & x^2z_1 - xz_1^2 \\ z_1^2t - z_1t^2 & z_1^2z_1 - z_1z_1^2 \end{vmatrix} dz_1$$

$$B_1(x,t) = -xt\left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5}\right)$$

Also, $B_2(x,t) = 0$

Hence, $B_p(x,t) = 0$ for all $p \geq 2$.

Next, $$\mu_1 = \int_0^1 K(z_1,z_1)dz_1 = \int_0^1 (z_1^3 - z_1^3)dz_1 = 0$$

$$\mu_2 = \int_0^1\int_0^1 \begin{vmatrix} K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1dz_2$$

$$\mu_2 = \int_0^1\int_0^1 \begin{vmatrix} 0 & z_1^2z_2 - z_1z_2^2 \\ z_2^2z_1 - z_2z_1^2 & z_2^2z_2 - z_2z_2^2 \end{vmatrix} dz_1dz_2$$

$$\mu_2 = \frac{1}{120}, \text{ (Upon simplification).}$$

Also, we find $\mu_3 = 0$.

Hence, $\mu_p = 0$ for all $p \geq 3$.

Now, we have

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t)$$

$$D(x,t;\lambda) = K(x,t) - \lambda B_1(x,t) + \cdots$$

$$D(x,t;\lambda) = x^2 t - xt^2 + \lambda xt\left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5}\right)$$

and
$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m = 1 - \lambda\mu_1 + \frac{\lambda^2}{2!}\mu_2 - \cdots$$

$$D(\lambda) = 1 + \frac{\lambda^2}{240}$$

Finally, $$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \frac{x^2 t - xt^2 + \lambda xt\left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5}\right)}{1 + (\lambda^2/240)}$$

EXAMPLE 7.3: Using the recurrence relations, find the resolvent kernels of the following kernels:

(a) $K(x,t) = x - 2t$; $0 \leq x \leq 1$, $0 \leq t \leq 1$

(b) $K(x,t) = \sin x \cos t$; $0 \leq x \leq 2\pi$, $0 \leq t \leq 2\pi$

(c) $K(x,t) = 4xt - x^2$; $0 \leq x \leq 1$, $0 \leq t \leq 1$

Solution: (a) Here, $K(x,t) = x - 2t$.

The resolvent kernel $R(x,t;\lambda)$ is given by Eq. (7.4) as

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} \tag{i}$$

where by Eq. (7.5), $$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) \tag{ii}$$

and by Eq. (7.6), $$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m \tag{iii}$$

Now, we have $B_0(x,t) = K(x,t) = x - 2t$ (iv)

Also, from Eqs. (7.43) and (7.44),

$$\mu_0 = 1 \quad \text{and} \quad \mu_p = \int_0^1 B_{p-1}(s,s)\,ds\,,\ p \geq 1 \tag{v}$$

and $$B_p(x,t) = \mu_p K(x,t) - p\int_0^1 K(x,z)B_{p-1}(z,t)\,dz\ ,\ p \geq 1 \qquad \text{(vi)}$$

Letting $p = 1$ in Eq. (v), we obtain

$$\mu_1 = \int_0^1 B_0(s,s)\,ds = \int_0^1 (s-2s)ds = -\frac{1}{2}$$

Let $p = 1$ in Eq. (vi). We obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z)B(z,t)dz$$

$$B_1(x,t) = -\frac{1}{2}(x-2t) - \int_0^1 (x-2z)(z-2t)\,dz$$

$$B_1(x,t) = -\frac{1}{2}(x-2t) - \int_0^1 \{-2z^2 + z(x+4t) - 2xt\}\,dz$$

$$B_1(x,t) = -\frac{1}{2}(x-2t) - \left[\frac{-2z^3}{3} + \frac{z^2}{2}(x+4t) - 2xtz\right]_0^1$$

$$B_1(x,t) = -\frac{1}{2}(x-2t) - \left[-\frac{2}{3} + \frac{1}{2}(x+4t) - 2xt\right]$$

$$B_1(x,t) = \frac{2}{3} + 2xt - x - t$$

Now, taking $p = 2$ in Eq. (v), we obtain

$$\mu_2 = \int_0^1 B_1(s,s)\,ds = \int_0^1 \left(\frac{2}{3} + 2s^2 - 2s\right)ds$$

$$\mu_2 = \left[\frac{2}{3}s + \frac{2s^3}{3} - s^2\right]_0^1 = \frac{1}{3}$$

Next, taking $p = 2$ in Eq. (vi), we obtain

and $B_2(x,t) = \mu_2 K(x,t) - 2\int_0^1 K(x,z)B_1(z,t)\,dz$

$$B_2(x,t) = \frac{1}{3}(x-2t) - 2\int_0^1 (x-2z)\left(\frac{2}{3} + 2zt - z - t\right)dz$$

$$B_2(x,t) = \frac{1}{3}(x-2t) - 2\int_0^1 \left[\frac{2x}{3} + 2xzt - xz - xt - \frac{4}{3}z - 4z^2t + 2z^2 + 2zt\right]dz$$

$$B_2(x,t) = \frac{1}{3}(x-2t) - 2\left[\frac{2xz}{3} + xtz^2 - \frac{xz^2}{2} - xtz - \frac{2}{3}z^2 - \frac{4}{3}z^3 t + \frac{2}{3}z^3 + z^2 t\right]_0^1$$

$$B_2(x,t) = \frac{1}{3}(x-2t) - \frac{4x}{3} - 2xt + x + 2xt + \frac{4}{3} + \frac{8}{3}t - \frac{4}{3} - 2t$$

$$B_2(x,t) = 0$$

Since $B_2(x,t) = 0$, therefore by Eq. (vi),

$B_p(x,t) = 0$ and $\mu_p = 0$ for all $p \geq 3$.

Substituting the above values in Eq. (ii) and (iii), we have

$$D(x,t;\lambda) = K(x,t) - \lambda B_1(x,t) + \cdots$$

$$D(x,t;\lambda) = x - 2t - \lambda\left(\frac{2}{3} + 2xt - x - t)\right)$$

and $$D(\lambda) = 1 - \lambda\mu_1 + \frac{\lambda^2}{2!}\mu_2 - \cdots$$

$$D(\lambda) = 1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}$$

Thus, by Eq. (i), $R(x,t;\lambda) = \dfrac{x - 2t - \lambda\left(\dfrac{2}{3} + 2xt - x - t\right)}{1 + \dfrac{\lambda}{2} + \dfrac{\lambda^2}{6}}$

(b) Here, $K(x,t) = \sin x \cos t$.

Now, $B_0(x,t) = K(x,t) = \sin x \cos t$

Also, $$\mu_0 = 1,\ \mu_p = \int_0^{2\pi} B_{p-1}(s,s)ds,\ p \geq 1 \tag{vii}$$

and $$B_p(x,t) = \mu_p K(x,t) - p\int_0^{2\pi} K(x,z)B_{p-1}(z,t)dz,\ p \geq 1 \tag{viii}$$

Let $p = 1$, in Eq. (vii). We have

$$\mu_1 = \int_0^{2\pi} B_0(s,s)ds = \int_0^{2\pi} \sin s \cos s\, ds$$

$$\mu_1 = \frac{1}{2}\int_0^{2\pi} \sin 2s\, ds = \frac{1}{2}\left[\frac{-\cos 2s}{2}\right]_0^{2\pi} = 0$$

Now, taking $p = 1$ in Eq. (viii), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^{2\pi} K(x,z) B_0(z,t)\, dz$$

$$B_1(x,t) = -\int_0^{2\pi} (\sin x \cos z)(\sin z \cos t) dz$$

$$B_1(x,t) = -\sin x \cos t \int_0^{2\pi} \sin z \cos z \; dz = 0$$

Since $B_1(x,t) = 0$, therefore by recurrence relations,

$$B_p(x,t) = 0 \quad \text{and} \quad \mu_p = 0 \text{ for all } p \geq 2.$$

Substituting these values in Eq. (ii) and (iii), we have

$$D(x,t;\lambda) = K(x,t) = \sin x \cos t$$

and $$D(\lambda) = 1$$

Hence, $$R(x,t;\lambda) = \sin x \, \cos t$$

(c) Here, $$K(x,t) = 4xt - x^2.$$

We have $$B_0(x,t) = K(x,t) = 4xt - x^2$$

$$\mu_0 = 1 \text{ and } \mu_p = \int_0^1 B_{p-1}(s,s) ds, \; p \geq 1 \tag{ix}$$

and $$B_p(x,t) = \mu_p K(x,t) - p\int_0^1 K(x,z) B_{p-1}(z,t) dz, \; p \geq 1 \tag{x}$$

Taking $p = 1$ in Eq. (ix), we obtain

$$\mu_1 = \int_0^1 B_0(s,s) ds = \int_0^1 (4s^2 - s^2) ds = [s^3]_0^1 = 1$$

Now, taking $p = 1$ in Eq. (x),we obtain

$$B_1(x,t) = \mu_1 \; K(x,t) - \int_0^1 K(x,z) B_0(z,t) dz$$

$$B_1(x,t) = 4xt - x^2 - \int_0^1 (4xz - x^2)(4zt - z^2)\, dz$$

$$B_1(x,t) = 4xt - x^2 - \left[-xz^4 + \frac{z^3}{3}(x^2 + 16xt) - 2x^2 tz \right]_0^1$$

$$B_1(x,t) = 4xt - x^2 - \left[-x + \frac{1}{3}(x^2 + 16xt) - 2x^2 t \right]$$

$$B_1(x,t) = 2x^2 t - \frac{4}{3}x^2 + x - \frac{4}{3}xt$$

Next taking, $p = 2$ in Eq. (ix), we obtain

$$\mu_2 = \int_0^1 B_1(s,s)ds = \int_0^1 \left(2s^3 - \frac{4}{3}s^2 + s - \frac{4}{3}s^2\right)ds$$

$$\mu_2 = \left[\frac{s^4}{2} - \frac{4}{9}s^3 + \frac{s^2}{2} - \frac{4s^3}{9}\right]_0^1 = \frac{1}{9}$$

and now putting $p = 2$ in Eq. (x), we obtain

$$B_2(x,t) = \mu_2 K(x,t) - 2\int_0^1 K(x,z)B_1(z,t)\,dz$$

$$B_2(x,t) = \frac{1}{9}(4xt - x^2) - 2\int_0^1 (4xz - x^2)\left(2z^2t - \frac{4}{3}z^2 + z - \frac{4}{3}zt\right)dz$$

$$B_2(x,t) = \frac{1}{9}(4xt - x^2) - 2\int_0^1 (4xz - x^2)\left[z^2\left(2t - \frac{4}{3}\right) + z\left(1 - \frac{4t}{3}\right)\right]dz$$

$$B_2(x,t) = \frac{1}{9}(4xt - x^2) - 2\int_0^1 \left[4x\left(2t - \frac{4}{3}\right)z^3 + z^2\left\{4x\left(1 - \frac{4t}{3}\right) - x^2\left(2t - \frac{4}{3}\right)\right\} - x^2\left(1 - \frac{4t}{3}\right)z\right]dz$$

$$B_2(x,t) = \frac{1}{9}(4xt - x^2) - 2\left[x\left(2t - \frac{4}{3}\right)z^4 + \frac{z^3}{3}\left\{4x\left(1 - \frac{4t}{3}\right) - x^2\left(2t - \frac{4}{3}\right)\right\} - \frac{x^2}{2}\left(1 - \frac{4t}{3}\right)z^2\right]_0^1$$

$$B_2(x,t) = \frac{1}{9}(4xt - x^2) - 2\left[x\left(2t - \frac{4}{3}\right) + \frac{1}{3}\left\{4x\left(1 - \frac{4t}{3}\right) - x^2\left(2t - \frac{4}{3}\right)\right\} - \frac{x^2}{2}\left(1 - \frac{4t}{3}\right)\right]$$

$B_2(x,t) = 0$

Since $B_2(x,t) = 0$, therefore $B_p(x,t) = 0$ and $\mu_p = 0$ for all $p \geq 2$.

Substituting these values in Eq. (ii) and (iii), we have

$$D(x,t;\lambda) = K(x,t) - \lambda B_1(x,t) = 4xt - x^2 - \lambda\left(2x^2t - \frac{4}{3}x^2 + x - \frac{4}{3}xt\right)$$

and $D(\lambda) = 1 - \lambda\mu_1 + \dfrac{\lambda^2}{2!}\mu_2 = 1 - \lambda + \dfrac{\lambda^2}{18}$

and thus, $R(x,t;\lambda) = \dfrac{4xt - x^2 - \lambda\left(2x^2t - \frac{4}{3}x^2 + x - \frac{4}{3}xt\right)}{1 - \lambda + (\lambda^2/18)}$

EXAMPLE 7.4: Determine $D(\lambda)$ and $D(x,t;\lambda)$ for the following kernels for the prescsibed limits a and b:

(a) $K(x,t) = 1$; $a = 0$, $b = 1$

(b) $K(x,t) = \sin x$; $a = 0$, $b = \pi$

Solution: (a) Here, $K(x,t) = 1$.
We know that

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) \tag{i}$$

and
$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m \tag{ii}$$

$$B_0(x,t) = K(x,t) = 1 \tag{iii}$$

$$\mu_0 = 0 \text{ and } \mu_p = \int_0^1 B_{p-1}(s,s)\, ds,\ p \geq 1 \tag{iv}$$

and
$$B_p(x,t) = \mu_p K(x,t) - p\int_0^1 K(x,z) B_{p-1}(z,t) dz,\ p \geq 1 \tag{v}$$

Taking $p = 1$ in Eq. (iv), we obtain

$$\mu_1 = \int_0^1 B_0(s,s) ds = \int_0^1 ds = 1$$

and taking $p = 1$ in Eq. (v), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z) B_0(z,t) dz$$

$$B_1(x,t) = 1 - \int_0^1 dz = 1 - 1 = 0$$

Since $B_1(x,t) = 0$, therefore $B_p(x,t) = 0$ and $\mu_p = 0$ for all $p \geq 2$.
Substituting the above values in Eq. (i) and (ii), we obtain

$$D(x,t;\lambda) = K(x,t) = 1$$

and
$$D(\lambda) = 1 - \lambda$$

(b) Here, $K(x,t) = \sin x$, Now, following the approach of part (a), we have

$$B_0(x,t) = K(x,t) = \sin x.$$

Again,
$$\mu_0 = 0 \text{ and } \mu_p = \int_0^{\pi} B_{p-1}(s,s)\, ds, p \geq 1 \tag{vi}$$

and
$$B_p(x,t) = \mu_p K(x,t) - p\int_0^{\pi} K(x,z) B_{p-1}(z,t)\, dz,\ p \geq 1 \tag{vii}$$

Taking $p = 1$ in Eq. (vi), we obtain

$$\mu_1 = \int_0^{\pi} B_0(s,s)\, ds = \int_0^{\pi} \sin s\, ds = [-\cos s]_0^{\pi} = 2$$

Taking $p = 1$ in Eq. (vii), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^{\pi} K(x,z) B_0(z,t)\, dz$$

$$B_1(x,t) = 2\sin x - \int_0^{\pi} \sin x \sin z\, dz$$

$$B_1(x,t) = 2\sin x - \sin x[-\cos z]_0^{\pi}$$

$$B_1(x,t) = 2\sin x - 2\sin x = 0$$

Since $B_1(x,t) = 0$, therefore by recursive nature,

$$B_p(x,t) = 0 \text{ and } \mu_p = 0 \text{ for all } p \geq 2.$$

Substituting these values in Eq. (ii) and (iii), we get

$$D(x,t;\lambda) = K(x,t) = \sin x, \text{ and } D(\lambda) = 1 - \lambda\mu_1 = 1 - 2\lambda$$

EXAMPLE 7.5: Using Fredholm theory, solve the following integral equations:

(a) $g(x) = e^x + \lambda \int_0^{10} xt\, g(t)\, dt$

(b) $g(x) = f(x) + \lambda \int_0^{1} (x+t)\, g(t)\, dt$

Solution: (a) Given $g(x) = e^x + \lambda \int_0^{10} xt\, g(t)\, dt$ (i)

Comparing Eq. (i) with

$$g(x) = f(x) + \lambda \int_0^{10} K(x,t)\, g(t)\, dt$$

we get $f(x) = e^x$ and $K(x,t) = xt$

We know that $B_0(x,t) = K(x,t) = xt$

$$\mu_0 = 1 \text{ and } \mu_p = \int_0^{10} B_{p-1}(s,s)\, ds, p \geq 1 \tag{ii}$$

and

$$B_p(x,t) = \mu_p K(x,t) - p \int_0^{10} K(x,z) B_{p-1}(z,t)\, dz, p \geq 1 \tag{iii}$$

Taking $p = 1$ in Eq. (ii), we obtain

$$\mu_1 = \int_0^{10} B_0(s,s)\, ds = \int_0^{10} s^2 ds = \left[\frac{s^3}{3}\right]_0^{10} = \frac{10^3}{3}$$

Now, taking $p = 1$ in Eq. (iii), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z) B_0(z,t)\,dz$$

$$B_1(x,t) = \frac{10^3}{3}xt - \int_0^{10} (xz)(zt)\,dz$$

$$B_1(x,t) = \frac{10^3}{3}xt - (xt)\left[\frac{z^3}{3}\right]_0^{10} = 0$$

Since $B_1(x,t) = 0$, therefore $B_p(x,t) = 0$, $\mu_1 = 0$ for all $p \geq 2$. Substituting these values in Eqs. (7.38) and (7.39), we have

$$D(x,t;\lambda) = K(x,t) = xt$$

$$D(\lambda) = 1 - \lambda\mu_1 = 1 - \lambda\left(\frac{10^3}{3}\right)$$

$$R\,(x,t;\lambda) = \frac{xt}{1 - \lambda(10^3/3)}$$

Thus, the required solution of Eq. (i) is given by

$$g(x) = f(x) + \lambda\int_0^{10} R(x,t;\lambda) f(t)\,dt$$

$$g(x) = e^x + \frac{\lambda x}{1 - \lambda(10^3/3)}\int_0^{10} te^t\,dt$$

$$g(x) = e^x + \frac{\lambda x}{1 - \lambda(10^3/3)}[te^t - e^t]_0^{10} \quad \text{(Integrating by parts)}$$

$$g(x) = e^x + \frac{3\lambda x}{3 - 1000\lambda}(10e^{10} - e^{10} + 1) \quad \text{(Upon simplification)}$$

$$g(x) = e^x + \frac{3\lambda x}{3 - 1000\lambda}(1 + 9e^{10})$$

(b) Here, we have $a = 0$, $b = 1$ and $K(x,t) = x + t$

Again, $B_0(x,t) = K(x,t) = x + t$

$$\mu_0 = 1 \text{ and } \mu_p = \int_0^1 B_{p-1}(s,s)\,ds,\ p \geq 1 \tag{iv}$$

and

$$B_p(x,t) = \mu_p K(x,t) - p\int_0^1 K(x,z) B_{p-1}(z,t)\,dz \tag{v}$$

For $p = 1$, Eq. (iv) gives

$$\mu_1 = \int_0^1 B_0(s,s)ds = \int_0^1 2s\,ds = [s^2]_0^1 = 1$$

Again, for $p = 1$, Eq. (v) provides

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z)B_0(z,t)\,dz$$

$$B_1(x,t) = (x+t) - \int_0^1 (x+z)(z+t)\,dz$$

$$B_1(x,t) = (x+t) - \int_0^1 [z^2 + z(x+t) + xt]\,dz$$

$$B_1(x,t) = (x+t) - \left[\frac{1}{3}z^3 + \frac{z^2}{2}(x+t) + xtz\right]_0^1$$

$$B_1(x,t) = (x+t) - \frac{1}{3} - \frac{1}{2}(x+t) - xt$$

$$B_1(x,t) = \frac{1}{2}(x+t) - xt - \frac{1}{3}$$

Also, putting $p = 2$ in Eq. (iv), we obtain

$$\mu_2 = \int_0^1 B_1(s,s)\,ds = \int_0^1 \left[\frac{1}{2}(s+s) - s^2 - \frac{1}{3}\right]ds$$

$$\mu_2 = \left[\frac{s^2}{2} - \frac{s^3}{3} - \frac{1}{3}s\right]_0^1 = -\frac{1}{6}$$

Now, puttng $p = 2$ in Eq. (v), we obtain

$$B_2(x,t) = \mu_2 K(x,t) - 2\int_0^1 K(x,z)B_1(z,t)dz$$

$$B_2(x,t) = -\frac{1}{6}(x+t) - 2\int_0^1 (x+z)\left[\frac{1}{2}(z+t) - zt - \frac{1}{3}\right]dz$$

$$B_2(x,t) = -\frac{1}{6}(x+t) - 2\int_0^1 \left[z^2\left(\frac{1}{2} - t\right) + z\left(\frac{1}{2}t - \frac{1}{3} + \frac{x}{2} - xt\right) + x\left(\frac{1}{2}t - \frac{1}{3}\right)\right]dz$$

$$B_2(x,t) = -\frac{1}{6}(x+t) - 2\left[\frac{z^3}{3}\left(\frac{1}{2} - t\right) + \frac{z^2}{2}\left(\frac{1}{2}t - \frac{1}{3} + \frac{x}{2} - xt\right) + xz\left(\frac{t}{2} - \frac{1}{3}\right)\right]_0^1$$

$$B_2(x,t) = -\frac{1}{6}(x+t) - 2\left[\frac{1}{3}\left(\frac{1}{2} - t\right) + \frac{1}{2}\left(\frac{1}{2}t - \frac{1}{3} + \frac{x}{2} - xt\right) + x\left(\frac{t}{2} - \frac{1}{3}\right)\right]$$

$B_2(x,t) = 0$ (Upon simplification)

Since $B_2(x,t) = 0$, it follows from Eqs. (iv) and (v), that

$$B_p(x,t) = 0,\ \mu_0 = 0 \text{ for all } p \geq 3$$

Therefore, $D(x,t;\lambda) = K(x,t) - \lambda B_1(x,t)$

$$D(x,t;\lambda) = x + t - \lambda\left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right]$$

and $$D(\lambda) = 1 - \lambda\mu_1 + \frac{\lambda^2}{2!}\mu_2 = 1 - \lambda - \frac{1}{12}\lambda^2$$

and thus, $$R\,(x,t;\lambda) = \frac{x + t - \lambda\left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right]}{1 - \lambda - (\lambda^2/12)}$$

Hence, the solution of the integral equation is given by

$$g(x) = f(x) + \lambda\int_0^1 R(x,t;\lambda)f(t)dt$$

or $$g(x) = f(x) + \lambda\int_0^1 \frac{(x+t) - \lambda\left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right]}{1 - \lambda - (\lambda^2/12)} f(t)dt.$$

EXAMPLE 7.6: Solve the following integral equations:

(a) $g(x) = x + \lambda\int_0^1 [xt + (xt)^{1/2}]g(t)dt$

(b) $g(x) = \dfrac{x}{6} + \lambda\int_0^1 (2x - t)g(t)\,dt$

Solution: (a) The given integral equation is

$$g(x) = x + \lambda\int_0^1 [xt + (xt)^{1/2}]\,g(t)\,dt \qquad \text{(i)}$$

Comparing it with

$$g(x) = f(x) + \lambda\int_a^b K(x,t)\,g(t)\,dt$$

we find, $a = 0$, $b = 1$, $f(x) = x$ and $K(x,t) = xt + (xt)^{1/2}$
so that here we have

$$B_0(x,t) = K(x,t) = xt + (xt)^{1/2}, \mu_0 = 1$$

and $$\mu_1 = \int_0^1 B_0(s,s)\,ds = \int_0^1 (s^2 + s)\,ds = \left[\frac{s^3}{3} + \frac{s^2}{2}\right]_0^1 = \frac{5}{6}$$

Next, $$B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z) B_0(z,t)\,dz$$

$$B_1(x,t) = \frac{5}{6}\{xt + (xt)^{1/2}\} - \int_0^1 \{xz + (xz)^{1/2}\}\{zt + (zt)^{1/2}\}dz$$

$$B_1(x,t) = \frac{5}{6}\{xt + (xt)^{1/2}\} - \left[\frac{xtz^3}{3} + (x\sqrt{t} + t\sqrt{x})\frac{z^{5/2}}{5/2} + \frac{z^2}{2}(xt)^{1/2}\right]_0^1$$

$$B_1(x,t) = \frac{5}{6}\{xt + (xt)^{1/2}\} - \left[\frac{xt}{3} + \frac{2}{5}(x\sqrt{t} + t\sqrt{x}) + \frac{1}{2}(xt)^{1/2}\right]$$

$$B_1(x,t) = \frac{1}{2}xt + \frac{1}{3}(xt)^{1/2} - \frac{2}{5}(x\sqrt{t} + t\sqrt{x})$$

Then, $$\mu_2 = \int_0^1 B_1(s,s)\,ds$$

$$\mu_2 = \int_0^1 \left[\frac{1}{2}s^2 + \frac{1}{3}s - \frac{2}{5}(s\sqrt{s} + s\sqrt{s})\right]ds$$

$$\mu_2 = \int_0^1 \left(\frac{1}{2}s^2 + \frac{1}{3}s - \frac{4}{5}s^{3/2}\right)ds$$

$$\mu_2 = \left[\frac{s^3}{6} + \frac{s^2}{6} - \frac{4}{5}\times\frac{s^{5/2}}{5/2}\right]_0^1 = \frac{1}{75}$$

Further, $$B_2(x,t) = \mu_2 K(x,t) - 2\int_0^1 (K(x,z) B_1(z,t)\,dz$$

$$B_2(x,t) = \frac{1}{75}\{xt + (xt)^{1/2}\} - 2\int_0^1 \{xz + (xz)^{1/2}\}\left\{\frac{1}{2}zt + \frac{1}{3}(zt)^{1/2} - \frac{2}{5}(z\sqrt{t} + t\sqrt{z})\right\}dz$$

$$B_2(x,t) = \frac{1}{75}\{xt + (xt)^{1/2}\} - 2\left[\frac{xtz^3}{6} + \frac{2x\sqrt{t}z^{5/2}}{15} - \frac{2x\sqrt{t}z^3}{15} - \frac{4xtz^{5/2}}{25} + \frac{t\sqrt{x}z^{5/2}}{5}\right.$$
$$\left. + \frac{(xt)^{1/2}z^2}{6} - \frac{4(xt)^{1/2}}{25}z^{5/2} - \frac{t\sqrt{x}\,z^2}{5}\right]_0^1$$

$$B_2(x,t) = \frac{1}{75}\{xt + (xt)^{1/2}\} - 2\left[\frac{xt}{6} + \frac{2x\sqrt{t}}{15} - \frac{2x\sqrt{t}}{15} - \frac{4xt}{25} + \frac{t\sqrt{x}}{5} + \frac{(xt)^{1/2}}{6} - \left(\frac{4}{25}\right)(xt)^{1/2} - t\sqrt{x}\right]$$

$B_2(x,t) = 0$ (Upon simplification)

Since $B_2(x,t) = 0$, therefore we infer that

$$B_p(x,t) = 0,\ \mu_p = 0 \text{ for all } p \geq 3$$

Thus, we find that

$$D(x,t;\lambda) = K(x,t) - \lambda B_1(x,t)$$

$$D(x,t;\lambda) = xt + (xt)^{1/2} - \lambda\left\{\frac{1}{2}xt + \frac{1}{3}(xt)^{1/2} - \frac{1}{2}(x\sqrt{t} + t\sqrt{x})\right\}$$

and

$$D(\lambda) = 1 - \lambda c_1 + \frac{\lambda^2}{2!}c_2 = 1 - \frac{5}{6}\lambda + \frac{1}{150}\lambda^2$$

Therefore, the resolvent kernel is given by

$$R(x,t;\lambda) = \frac{xt + (xt)^{1/2} - \lambda\left\{\frac{1}{2}xt + \frac{1}{3}(xt)^{1/2} - \frac{2}{5}(x\sqrt{t} + t\sqrt{x}\right\}}{1 - (5/6)\lambda + (1/150)\lambda^2}$$

Hence, the required solution of the given equation is

$$g(x) = f(x) + \lambda\int_0^1 R(x,t;\lambda)f(t)\,dt$$

or $$g(x) = x + \lambda\int_0^1 \frac{xt + (xt)^{1/2} - \lambda\left\{\frac{1}{2}xt + \frac{1}{3}(xt)^{1/2} - \frac{2}{5}(x\sqrt{t} + t\sqrt{x})\right\}}{1 - (5/6)\lambda + (1/150)\lambda^2}dt$$

$$g(x) = x + \frac{\lambda}{1 - (5/6)\lambda + (1/150)\lambda^2}$$

$$\left[\frac{xt^3}{3} + \frac{2\sqrt{x}t^{5/2}}{5} - \frac{\lambda xt^3}{6} - \frac{2\sqrt{x}\lambda t^{5/2}}{15} - \frac{4x\lambda t^{5/2}}{25} - \frac{2\sqrt{x}\lambda t^3}{15}\right]_0^1$$

$$g(x) = x + \frac{\lambda}{1 - (5/6)\lambda + (1/150)\lambda^2}\left[\frac{x}{3} + \frac{2\sqrt{x}}{5} - \frac{\lambda x}{6} - \frac{2\lambda\sqrt{x}}{15} - \frac{4x\lambda}{25} - \frac{2\lambda\sqrt{x}}{15}\right]$$

$$g(x) = \frac{150x + \lambda(60\sqrt{x} - 75x) + 21x\lambda^2}{\lambda^2 - 125\lambda + 150}$$

(b) Here, $f(x) = x/6$, $K(x,t) = 2x - t$.

Proceeding as in Example 7.1(b), we obtain

$$R(x,t;\lambda) = \frac{2x - t - \lambda\left(\frac{2}{3} - x - t + 2xt\right)}{1 - (\lambda/2) + (\lambda^2/6)}$$

Hence, the required solution by Eq. (7.36) is

$$g(x) = f(x) + \lambda \int_0^1 R(x,t;\lambda) f(t)\, dt$$

or
$$g(x) = \frac{x}{6} + \lambda \int_0^1 \left\{ \frac{2x - t - \lambda\left(\frac{2}{3} - x - t + 2xt\right)}{1 - (\lambda/2) + (\lambda^2/6)} \right\} \frac{t}{6}\, dt$$

$$g(x) = \frac{x}{6} + \frac{\lambda}{\lambda^2 - 3\lambda + 6} \int_0^1 \left\{ 2x - t^2 - \lambda\left(\frac{2}{3}t - xt - t^2 + 2xt\right) \right\} dt$$

$$g(x) = \frac{x}{6} + \frac{\lambda}{\lambda^2 - 3\lambda + 6} \left[xt^2 - \frac{t^3}{3} - \lambda\left(\frac{1}{3}t^2 - \frac{xt^2}{2} - \frac{t^3}{3} + \frac{xt^3}{3}\right) \right]_0^1$$

$$g(x) = \frac{x}{6} + \frac{\lambda}{\lambda^2 - 3\lambda + 6} \left[x - \frac{1}{3} - \lambda\left(\frac{1}{3} - \frac{x}{2} - \frac{1}{3} + \frac{x}{3}\right) \right]$$

$$g(x) = \frac{x}{6} + \frac{\lambda}{\lambda^2 - 3\lambda + 6} \left[x - \frac{1}{3} + \frac{\lambda x}{6} \right]$$

$$g(x) = \frac{1}{6} \left[x + \frac{\lambda}{\lambda^2 - 3\lambda + 6} (6x - 2 + \lambda x) \right]$$

$$g(x) = \frac{1}{6} \left[x + \frac{\lambda(6x - 2) + \lambda^2 x}{\lambda^2 - 3\lambda + 6} \right]$$

7.4 FREDHOLM'S SECOND FUNDAMENTAL THEOREM

Fredholm's first theorem does not hold good when λ is a root of the equation $D(\lambda) = 0$. It has been found in Chapter 3 that for a separable kernel, the homogeneous equation

$$g(x) = \lambda \int_a^b K(x,t) g(t)\, dt \tag{7.45}$$

has non-trivial solutions. It is expected that same holds when the kernel is an arbitrary integrable function and we shall then have a spectrum of eigenvalues and the corresponding eigenfunctions. This second Fredholm theorem is for the study of this problem.

Statement: If λ_0 is a zero of multiplicity m of the function $D(\lambda)$, then the homogeneous equation [Eq. (7.45)] possesses at least one, and at most m, linearly independent solutions given by

$$g_i(x) = D_r \left(\begin{matrix} x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_r \\ t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_r \end{matrix} \middle| \lambda_0 \right); \quad (i = 1, 2, \ldots, r, 1 \le r \le m) \tag{7.46}$$

Also, any other solution is a linear combination of these solutions. It is to be recollected that the definition of the fredholm minor

$$D_n\left(\begin{matrix} x_1, x_2, \ldots, x_n \\ t_1, t_2, \ldots, t_n \end{matrix}\middle| \lambda\right) = K\begin{pmatrix} x_1, x_2, \ldots, x_n \\ t_1, t_2, \ldots, t_n \end{pmatrix}$$

$$+\sum_{j=1}^{\infty}\frac{(-\lambda)^j}{j!}\int_a^b \ldots \int_a^b K\begin{pmatrix} x_1, \ldots, x_n, z_1, \ldots, z_j \\ t_1, \ldots, t_n, z_1, \ldots, z_j \end{pmatrix} dz_1 \ldots dz_j \tag{7.47}$$

where $\{x_i\}$ and $\{t_i\}$, $(i = 1,2,\ldots,n)$ are two sequences of arbitrary variables, Eq. (7.47) converges for all values of λ, and hence, it is an entire function of λ.

Proof: First, it will be proved that every zero of $D(\lambda)$ is a pole of the resolvent kernel $R(x,t;\lambda)$ given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} \tag{7.48}$$

The order of its pole is at most equal to the order of the zero of $D(\lambda)$.
The Fredholm's first series is given by

$$D(\lambda) = 1 + \sum_{j=1}^{\infty}\frac{(-\lambda)^j}{j!}\int_a^b \ldots \int_a^b K\begin{pmatrix} x_1, \ldots, x_j \\ x_1, \ldots, x_j \end{pmatrix} dx_1 \ldots dx_j \tag{7.49}$$

and the Fredholm's second series is given by

$$D(x,t;\lambda) = K(x,t) + \sum_{j=1}^{\infty}\frac{(-\lambda)^j}{j!}\int_a^b \ldots \int_a^b K\begin{pmatrix} x, x_1 \ldots, x_j \\ t, x_1, \ldots, x_j \end{pmatrix} dx_1 \ldots dx_j \tag{7.50}$$

Differentiating both sides of Eq. (7.49) with respect to λ and interchanging the indices of the variables of integration, it can be expressed as

$$D^1(\lambda) = -\int_a^b D(x,x;\lambda)\,dx \tag{7.51}$$

From this relation, it follows that if λ_0 is a zero of order p of $D(\lambda)$, then it is a zero of order $(p-1)$ of $D'(\lambda)$, and consequently, λ_0 may be a zero of order at most $(p-1)$ of the entire function $D(x,t;\lambda)$. Thus, λ_0 is the pole of order at most p, particularly, if λ_0 is a simple pole of $D(\lambda)$, then $D(\lambda_0) = 0$, $D'(\lambda_0) \neq 0$ and λ_0 is a simple pole of the resolvent kernel. Moreover, it follows from Eq. (7.51) that $D(x,t;\lambda) \neq 0$. In this particular case, it is observed from the following equation:

$$D(x,t;\lambda) = K(x,t)D(\lambda) + \lambda\int_a^b K(x,z)D(z,t;\lambda)\,dz$$

i.e., if $D(\lambda) = 0$ and $D(x,t;\lambda) \neq 0$, then $D(x,t;\lambda)$, as a function of x, is a solution of Eq. (7.45) and so is $\sigma D(x,t;\lambda)$**,** σ being an arbitrary constant.

We now consider the general case when λ is a zero of an arbitrary multiplicity m, i.e., when

$$D^{(r)}(\lambda_0) = 0 \text{ and } D^{(m)}(\lambda_0) \neq 0, \tag{7.52}$$

where $r = 1, 2, \ldots, m-1$.

Differentiating n times the Fredholm's first series, i.e., Eq. (7.49), and obtain

$$D^{(n)}(\lambda) = (-1)^n \int_a^b \ldots \int_a^b K\begin{pmatrix} x_1, x_2, \ldots, x_n \\ x_1, x_2, \ldots, x_n \end{pmatrix} dx_1 \ldots dx_n$$

$$+ (-1)^n \sum_{j=1}^{\infty} \frac{(-\lambda)^j}{j!} \int_a^b \ldots \int_a^b K\begin{pmatrix} x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+j} \\ x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+j} \end{pmatrix} dx_1\, dx_2 \ldots dx_{n+j} \tag{7.53}$$

Now comparing Eqs. (7.51) and (7.53), we have

$$D^{(n)}(\lambda) = (-1)^n \int_a^b \ldots \int_a^b D_n\left(\begin{matrix} x_1, x_2, \ldots, x_n \\ x_1, x_2, \ldots, x_n \end{matrix}\middle| \lambda\right) dx_1 \ldots dx_n \tag{7.54}$$

which is a relation between nth derivative of the Fredholm function and Fredholm minor of order n. From Eq. (7.54), it is concluded that if λ_0 is a zero of order m of the function $D(\lambda)$, then the following holds for the Fredholm minor of order m for that value of λ_0:

$$D_m\left(\begin{matrix} x_1, x_2, \ldots, x_m \\ t_1, t_2, \ldots, t_m \end{matrix}\middle| \lambda_0\right) \neq 0 \tag{7.55}$$

Thus, there may exist minors of order lower than m, and which also do not identically vanish.

We now establish the relation among the minors that corresponds to the resolvent formula

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,z) R(z,t;\lambda)\, dz$$

Expanding the determinant under the integral sign in Eq. (7.47),

$$\begin{vmatrix} K(x_1,t_1) & K(x_1,t_2)\ldots K(x_1,t_n) & K(x_1,z_1)\ldots K(x_1,z_j) \\ K(x_2,t_1) & K(x_2,t_2)\ldots K(x_2,t_n) & K(x_2,z_1)\ldots K(x_2,z_j) \\ \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \\ K(x_n,t_1) & K(x_n,t_2)\ldots K(x_n,t_n) & K(x_n,z_1)\ldots K(x_n,z_j) \\ K(z_1,t_1) & K(z_1,t_2)\ldots K(z_1,t_n) & K(z_1,z_1)\ldots K(z_1,z_j) \\ \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \\ K(z_j,t_1) & K(z_j,t_2)\ldots K(z_j,t_n) & K(z_j,z_1)\ldots K(z_j,z_j) \end{vmatrix} \tag{7.56}$$

in terms of the elements of the first row and integrating j times with respect to z_1, z_2, ..., z_j for $j \geq 1$, we obtain

$$\int_a^b \ldots \int_a^b K\begin{pmatrix} x_1,\ldots,x_n,z_1,\ldots,z_j \\ t_1,\ldots,t_n,z_1,\ldots,z_j \end{pmatrix} dz_1 \ldots dz_j = \sum_{l=1}^{n} (-1)^{l+1} K(x_1,t_n)$$

$$\times \int_a^b \ldots \int_a^b K\begin{pmatrix} x_2,\ldots,x_l,\ldots,x_n,z_1,\ldots,z_j \\ t_1,\ldots,t_{l-1},t_{l+1},\ldots,t_n,z_1,\ldots,z_j \end{pmatrix} dz_1 dz_2 \ldots dz_j + \sum_{l=1}^{p} (-1)^{l+n-1}$$

$$\times \int_a^b \ldots \int_a^b K(x_1,z_l) K\begin{pmatrix} x_2,\ldots,x_n,z_1,z_2,\ldots,z_l,\ldots,z_j \\ t_1,\ldots,t_{n-1},t_n,z_1,z_2,\ldots,z_{l-1},z_{l+1},\ldots,z_j \end{pmatrix} dz_1 \ldots dz_j \quad (7.57)$$

Here, it is noted that the symbols for the determinant K on the right-hand side of Eq. (7.57) do not contain the variable x_i in the upper sequence and the variables t_1 or z_l in the lower sequence. Further, by transposing the variable x_l in the upper sequence to the first place by means of $l + n - 2$ transpositions, it is followed that all the components of the second sum on the right side are equal. Therefore, Eq. (7.57) can be writtern as

$$\int_a^b \ldots \int_a^b K\begin{pmatrix} x_1,\ldots,x_n,z_1,\ldots,z_j \\ t_1,\ldots,t_n,z_1,\ldots,z_j \end{pmatrix} dz_1 \ldots dz_j = \sum_{l=1}^{n} (-1)^{l+1} K(x_1,t_l)$$

$$\times \int_a^b \ldots \int_a^b K\begin{pmatrix} x_2,\ldots,x_n,z_1,\ldots,z_j \\ t_1,\ldots,t_{l-1},t_{l+1},\ldots,t_n,z_1,\ldots,z_j \end{pmatrix} dz_1 \ldots dz_j$$

$$-j\int_a^b K(x_1,z) \left[\int_a^b \ldots \int_a^b K\begin{pmatrix} z,x_2,\ldots,x_n,z_1,\ldots,z_{j-1} \\ t_1,t_2,\ldots,t_n,z_1,\ldots,z_{j-1} \end{pmatrix} dz_1 \ldots dz_{j-1} \right] dz \quad (7.58)$$

where the subscript l from z has been omitted.

From Eqs. (7.57) and (7.58), we find that Fredholm minor satisfies the following integral equation:

$$D_n\left(\begin{matrix} x_1,x_2,\ldots,x_n \\ t_1,t_2,\ldots,t_n \end{matrix} \middle| \lambda \right) = \sum_{l=1}^{n} (-1)^{l+1} K(x_1,t_n) D_{n-1}\begin{pmatrix} x_2,\ldots,x_n \\ t_1,\ldots,t_{l-1},t_{l+1},\ldots,t_n \end{pmatrix}$$

$$+\lambda \int_a^b K(x_1,z) D_n\left(\begin{matrix} z,x_2,\ldots,x_n \\ t_1,t_2,\ldots,t_n \end{matrix} \middle| \lambda \right) dz \quad (7.59)$$

Expanding the determinant shown in Eq. (7.56) with respect to the first column and proceeding as above, we obtain the integral equation:

$$D_n\begin{pmatrix} x_1,\ldots,x_n \\ t_1,t,\ldots,t_n \end{pmatrix}\lambda\Bigg) = \sum_{l=1}^{n}(-1)^{l+1}K(x_1,t_n)D_{n-1}\begin{pmatrix} x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_n \\ t_2,\ldots\ldots\ldots\ldots,t_n \end{pmatrix}dz$$

$$+\lambda\int_a^b K(z,\ t_1)D_n\begin{pmatrix} x_1,\ldots,x_n \\ z,t_2,\ldots,t_n \end{pmatrix}dz \tag{7.60}$$

The relations shown in Eqs. (7.59) and (7.60) hold for all values of λ. With the help of (7.59), one can find the solution of Eq. (7.45) for the special case when $\lambda = \lambda_0$ is an eigenvalue of the kernel, by supposing that $\lambda = \lambda_0$ is a zero of multiplicity m of D(λ). Then, as mentioned earlier, the minor $D_m \neq 0$ and even the minors $D_1, D_2 \ldots, D_{m-1}$ may not identically vanish. Suppose D_r is the first minor in the sequence $D_1, D_2 \ldots, D_{m-1}$ such that $D_r \neq 0$. Then number r must be between 1 and m and is the index of eigenvalue λ_0. It thus follows that $D_{r-1} = 0$. Then Eq. (7.59) shows that

$$g_1(x) = D_r\left(\begin{matrix} x_1,x_2,\ldots,x_r \\ t_1,t_2,\ldots,t_r \end{matrix}\middle|\,\lambda_0\right) \tag{7.61}$$

is a solution of Eq. (7.45). Substituting x at different points of the upper sequence in the minor D_r, we obtain r non -trivial solutions $g_i(x)$, (i = 1,2, ..., r) of Eq. (7.45) which are often written as

$$\phi_i(x) = \frac{D\left(\begin{matrix} x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_r \\ t_1,\ldots\ldots\ldots\ldots\ldots,t_r \end{matrix}\middle|\,\lambda_0\right)}{D_r\left(\begin{matrix} x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_r \\ t_1,\ldots\ldots\ldots\ldots\ldots,t_r \end{matrix}\middle|\,\lambda_0\right)};\qquad (i = 1,2,\ldots,r) \tag{7.62}$$

in which the denominator is non–zero.

We now establish that solutions ϕ_i given by Eq. (7.62) are linearly independent. In the determinant shown in Eq. (7.56), if we put two of the arguments x_i equal, this is equivalent to having two rows equal, and consequently, the determinant vanishes. Thus in Eq. (7.62), we see that $\phi_k(x_i) = 0$ for $i \neq k$, whereas $\phi_k(x_k) = 1$.

Now, if we have a relation of the form $\sum_k c_k\phi_k = 0$, then putting $x = x_i$, we get $c_i = 0$ and so the solution ϕ_i are linearly independent. This system of solutions ϕ_i is known as fundamental system of eigenfunctions of λ_0, and any linear combination of these functions gives a solution of Eq. (7.45).

Conversely, it can be shown that any solution of Eq. (7.45) must be a linear combination of $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_v(x)$. For this purpose, we define a kernel $H(x,t;\lambda)$, which corresponds to the resolvent kernel $R(x,t;\lambda)$ of Section 7.2.

$$H(x,t;\lambda) = \frac{D_{r+1}\left(\begin{matrix} x,\ x_1,\ldots,x_r \\ t,t_1,\ldots,t_r \end{matrix}\middle|\,\lambda_0\right)}{D_r\left(\begin{matrix} x_1,\ldots,x_r \\ t_1,\ldots,t_r \end{matrix}\middle|\,\lambda_0\right)} \tag{7.63}$$

Putting $n = r$ and adding extra arguments x and t in Eq. (7.59), we obtain

$$D_{r+1}\left(\begin{matrix} x, x_1,\dots,x_r \\ t,t_1,\dots,t_r \end{matrix}\middle|\lambda_0\right) = K(x,t)D_{r+1}\left(\begin{matrix} x_1,\dots,x_r \\ t_1,\dots,t_r \end{matrix}\middle|\lambda_0\right) + \sum_{l=1}^{r}(-1)^l K(x_l,t)$$

$$D_r\left(\begin{matrix} x, x_1,\dots,x_{l-1},x_{l+1},\dots,x_r \\ t_1,t_2\dots,t_r \end{matrix}\middle|\lambda_0\right) + \lambda_0\int_a^b K(z,t)D_{r+1}\left(\begin{matrix} x,x_1,\dots,x_r \\ z,t_1,\dots,t_r \end{matrix}\middle|\lambda_0\right) \quad (7.64)$$

In each minor D_r, of Eq. (7.64), we transpose the variable x from the first place to the place between the variables x_{l-1} and x_{l+1} and divide both sides by the constant

$$D_r\left(\begin{matrix} x_1,\dots,x_r \\ t_1,\dots,t_r \end{matrix}\middle|\lambda_0\right) \neq 0$$

to get

$$H(x,t;\lambda) - K(x,t) - \lambda_0\int_a^b H(x,z;\lambda)K(z,t)dz = -\sum_{l=1}^{r} K(x_l,t)\,\phi_l(x) \quad (7.65)$$

Now suppose that $g(x)$ is an arbitrary solution of Eq. (7.45). Multiplying both sides of Eq. (7.65) by $g(t)$, and then, integrating both sides with respect to t from a to b, we get

$$\int_a^b g(t)H(x,t;\lambda)\,dt - \frac{g(x)}{\lambda_0} - \int_a^b g(z)H(x,z;\lambda)\,dz = -\sum_{l=1}^{r}\frac{g(x_l)}{\lambda_0}\phi_l(x) \quad (7.66)$$

where we have used Eq. (7.45) in all terms except the first. Further, we have also taken

$$\lambda_0\int_a^b K(x_l,t)g(t)dt = g(x_l)$$

Cancelling the equal terms in Eq. (7.66), we obtain

$$g(x) = \sum_{l=1}^{r} g(x_l)\,\phi_l(x)$$

It, thus, completes the proof.

7.5 FREDHOLM'S THIRD THEOREM

Statement: For the non-homogeneous integral equation of the second kind

$$g(x) = f(x) + \lambda_0\int_a^b K(x,t)g(t)dt \quad (7.67)$$

to have a solution in the case $D(\lambda_0) = 0$, it is necessary and sufficient that the given function *f(x)* is orthogonal to all the eigenfunctions $h_i(x)$, $i = 1,2,\dots, n$ of the transposed homogeneous equation corresponding to the eigenvalue λ_0, and then, the corresponding general solution has the form

$$g(x) = f(x) + \lambda \int_a^b \frac{D_{r+1}\left(\begin{matrix} x, x_1, x_2, ..., x_r \\ t, t_1, t_2, ..., t_r \end{matrix} \middle| \lambda_0 \right)}{D_r\left(\begin{matrix} x_1, x_2, ..., x_r \\ t_1, t_2, ..., t_r \end{matrix} \middle| \lambda_0 \right)} f(t)dt + \sum_{l=1}^{r} c_l \phi_l(x) \tag{7.68}$$

Proof: Consider $g(x) = f(x) + \lambda \int_a^b K(x,t)g(t)dt$.

Let the transpose (or adjoint) of Eq. (7.67) be

$$h(x) = f(x) + \lambda \int_a^b K(t,x)\, h(t)\, dt \tag{7.69}$$

Now, for the transposed Eq. (7.69), Fredholm's first and second series $D(\lambda)$ and $D(t,x;\lambda)$ are given by

$$D(\lambda) = 1 + \sum_{j=1}^{\infty} \frac{(-\lambda)^j}{j!} \int_a^b ... \int_a^b K\begin{pmatrix} z_1, ..., z_j \\ z_1, ..., z_j \end{pmatrix} dz_1 ....dz_j \tag{7.70}$$

and $$D(t,x;\lambda) = K(t,x) + \sum_{j=1}^{\infty} \frac{(-\lambda)^j}{j!} \int_a^b ... \int_a^b K\begin{pmatrix} t, z_1, ..., z_j \\ x, z_1, ..., z_j \end{pmatrix} dz_1 ....dz_j \tag{7.71}$$

respectively. From this fact, it follows that kernels of Eq. (7.67)and its transpose in Eq. (7.69) have the same eigenvalues. Further, the corresponding resolvent kernel for Eq. (7.69) is

$$R(t,x;\lambda) = D(t,x;\lambda)/D(\lambda) \tag{7.72}$$

and hence, the solution of Eq. (7.69) is

$$h(x) = f(x) + \lambda \int_a^b \frac{D(t,x;\lambda)}{D(\lambda)} f(t)\, dt \tag{7.73}$$

provided λ is not an eigenvalue.

Next, it is obvious that not only the transposed kernel has the eigenvalues as the original kernel of Eq. (7.67) but also the index r of each of the eigenvalues is equal. Moreover, the eigenfunctions of the transposed equation for an eigenvalue λ_0 are given by

$$h_i(t) = \frac{D_r\begin{pmatrix} x_1, ..., x_r \\ t_1, ..., t_{i-1}, t, t_{i+1}, ..., t_r \end{pmatrix}}{D_r\begin{pmatrix} x_1, ..., x_r \\ t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_r \end{pmatrix}} \tag{7.74}$$

where the values $(x_1, ..., x_r)$ and $(t_1, ..., t_r)$ are to be so chosen that the denominater does not vanish. Substituting r in different places in the lower sequence of

this formula, we obtain a linearly independent system of r eigenfunctions. Also, we know that each ϕ_i is orthogonal to each with different eigenvalues.

Suppose $g(x)$ is a solution of Eq. (7.67). Then, multiplying Eq. (7.67) by each member $h_p(x)$ of the above-mentioned system of functions and integrating with respect to x from a to b, we obtain

$$\int_a^b f(x)h_p(x)\,dx = \int_a^b g(x)h_p(x)\,dx - \lambda\int_a^b\int_a^b K(x,t)g(t)h_p(x)dx\,dt$$

$$\int_a^b f(x)h_p(x)\,dx == \int_a^b g(x)\left[h_p(x) - \lambda\int_a^b K(t,x)h_p(t)\,dt\right]dx \tag{7.75}$$

Since $h_p(x)$ is an eigenfunction of the transposed homogeneous equation, therefore

$$h_p(x) = \lambda\int_a^b K(t,x)h_p(t)\,dt \tag{7.76}$$

where λ is the corresponding eigenvalue. Now, from Eqs. (7.75) and (7.76), we have

$$\int_a^b f(x)h_p(x)\,dx = 0 \tag{7.77}$$

Thus, it follows that a necessary condition for Eq. (7.77) to have a solution is that the non-homogeneous term $f(x)$ be orthogonal to each solution of the transposed homogeneous equation.

Conversely, now it will be proved that the condition Eq. (7.77) of orthogonality is sufficent for the existence of a solution. In what follows, we shall also obtain an explicit solution in such a case. At this stage, we define a kernel $H(t,x;\lambda)$ as follows:

$$H(t,x;\lambda) = \frac{D_{r+1}\left(\begin{array}{c|c} x, x_1, \ldots, x_r & \lambda_0 \\ t, t_1, \ldots, t_r & \end{array}\right)}{D_r\left(\begin{array}{c|c} x_1, \ldots, x_r & \lambda_0 \\ t_1, \ldots, t_r & \end{array}\right)} \tag{7.78}$$

wherein it is assumed that $D_r \neq 0$ and that r is the index of eigenvalue λ_0. To prove the required result, we show that if the orthogonality condition is satisfied, then the function

$$g_0(x) = f(x) + \lambda_0\int_a^b H(x,t;\lambda)f(t)\,dt \tag{7.79}$$

is a solution.

Substituting this value for $g(x)$ in Eq. (7.67), we have [by referring to Eq. (7.65)]

$$f(x) + \lambda_0\int_a^b H(x,t;\lambda)f(t)\,dt = f(x) + \lambda_0\int_a^b K(x,t)\left[f(t) + \lambda_0\int_a^b H(t,z;\lambda)f(z)\,dz\right]dt$$

or $$\int_a^b f(t)\,dt\left[H(t,z;\lambda)-K(x,t)-\lambda_0\int_a^b K(x,z)H(z,t;\lambda)dz\right]=0 \tag{7.80}$$

Proceeding similarly, we obtain its transpose as

$$H(x,t;\lambda)-K(x,t)-\lambda_0\int_a^b K(x,z)H(z,t;\lambda)dz=-\sum_{p=1}^{r}K(x,t_p)h_p(t)$$

Substituting it in Eq. (7.80) and making use of the orthogonality condition, we have an identity, and thus, it is proved.

Here, the difference of any two solutions of Eq. (7.67) is a solution of the homogeneous equation. Hence, the most general solution of (7.67) is

$$g(x)=f(x)+\lambda_0\int_a^b H(x,t;\lambda)f(t)\,dt+\sum_{p=1}^{r}c_p\phi_p(x)$$

EXERCISE 7.1

1. Using the Fredholm determinants, find the resolvent kernel of the following kernels:

   (a) $K(x,t) = \sin x - \sin t$, $0 \le x \le 2\pi$, $0 \le t \le 2\pi$

   (b) $K(x,t) = 2e^x e^t$, $a = 0$, $b = 1$

2. Using the recursion relations, find the resolvent kernels of the following kernels:

   (a) $K(x,t) = x + t + 1$, $-1 \le x \le 1$, $-1 \le t \le 1$

   (b) $K(x,t) = \sin(x + t)$, $0 \le x \le 2\pi$, $0 \le t \le 2\pi$

3. For the integral equation

   $$g(x)=f(x)+\lambda\int_a^b K(x,t)g(t)\,dt$$

   compute $D(\lambda)$ and $D(x,t;\lambda)$ for the following kernels for the prescribed limits a and b:

   (a) $K(x,t) = \sin\,(x + t)$, $a = 0$, $b = \pi$

   (b) $K(x,t) = e^{x-t}$, $a = 0$, $b = 1$

4. Determine the resolvent kernel, and hence solve the following integral equations:

   (a) $$g(x)=e^x-\int_0^1 e^{x-t}g(t)\,dt$$

   (b) $$g(x)=\cos 2x+\int_0^{2\pi}\sin x\cos t\,g(t)\,dt$$

Answers:

1. (a) $R(x,t;\lambda) = \dfrac{\sin x - \sin t - \lambda\pi(1 + 2\sin x \sin t)}{1 + 4\pi^2\lambda^2}$

   (b) $R(x,t;\lambda) = \dfrac{2e^x e^t}{1 - \lambda(e^2 - 1)}$

2. (a) $R(x,t;\lambda) = \dfrac{x + t + 1 + 2\lambda(xt + 1/3)}{1 - 2\lambda - (4/3)\lambda^2}$

   (b) $R(x,t;\lambda) = \dfrac{\sin(x + t) + \pi\lambda\cos(x - t)}{1 - \pi^2\lambda^2}$

3. (a) $D(x,t;\lambda) = \sin(x + t) + (\lambda\pi/2)\cos(x - t), D(\lambda) = 1 - (\pi^2/4)\lambda^2$

   (b) $D(x,t;\lambda) = x, D(\lambda) = 1 - 42\lambda$

4. (a) $g(x) = \dfrac{1}{2}e^x$

   (b) $g(x) = \cos 2x$

❖❖❖

CHAPTER 8

Integral Transform Methods

8.1 INTRODUCTION

The integral transform methods provide a useful tool for the solution of integral equations of various special forms. Let the following double integral exist:

$$g(x) = \int_a^b \int_a^b F(x, t_1) K(t_1, t) g(t)\, dt\, dt_1 \tag{8.1}$$

This double integral can be evaluated as an iterated integral.

If we take
$$f(x) = \int_a^b K(x, t) g(t)\, dt, \tag{8.2}$$

then from Eq. (8.1), we have

$$g(x) = \int_a^b F(x, t) f(t)\, dt \tag{8.3}$$

Thus, if Eq. (8.1) is regarded as an integral equation in g, a solution is given by Eq. (8.2), whereas if Eq. (8.3) is regarded as an integal equation in f, a solution is given by Eq. (8.2). It is conventional to refer, one of these functions as the transform of the second function, and to the second function, as an inverse transform of the first.

Some examples* are given here.

1. The most well-known double integral Eq. (8.1) is the Fourier integral

$$g(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-isx} e^{-ixt} g(t) dt\, dx$$

which results in the reciprocal relations as:

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}\, g(t)\, dt$$

* Refer to Chapter 6 of *Integral Transforms* published by RBD.

and has
$$g(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist} f(t)\, dt$$

The function $f(s)$ is known as Fourier transform of $g(t)$ and $g(s)$ as the inverse Fourier transform of $f(t)$, and vice versa.

2. Considering the double integral $g(s)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}(\sin sx\ \sin xt)\,g(t)\,dx$, we find that this leads to the sine transform $f(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}(\sin st)\,g(t)\,dt$ and its inverse as, $g(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}(\sin st)\,f(t)\,dt$, respectively.

8.2 SINGULAR INTEGRAL EQUATION

Singular integral equations occur frequently in mathematical physics and possess very unusual properties; hence, their solutions are quite essential.

Definition: An integral equation is called *singular* if either the range of integration is infinite or the kernel is discontinuous.

For example, the singular integral equations of the first kind are:

$$f(x)=\int_0^{\infty}\sin(xt)\,g(t)\,dt \tag{8.4}$$

$$f(x)=\int_0^{\infty}e^{-xt}\,g(t)\,dt \tag{8.5}$$

In Eqs. (8.4) and (8.5), the range of integration is infinite, while in integral

$$f(x)=\int_0^{x}\frac{g(t)}{\sqrt{x-t}}\,dt \tag{8.6}$$

the range of integration is finite, but the kernel is discontinuous.

The Abel's Integral Equation

One of the simplest form of singular integral equation, which appears in mechanics, is the Abel's integral equation, given as,

$$f(x)=\int_0^{x}\frac{g(t)}{(x-t)^{\alpha}}\,dt,\qquad 0<\alpha<1 \tag{8.7}$$

where $g(t)$ is an unknown function to be determined and $f(x)$ is a known function.

8.3 LAPLACE TRANSFORM*

Definition: Let $f(x)$ be a function defined for $x > 0$. Then, the Laplace transform of $f(x)$, denoted by $L\{f(x); p\}$ or $F(p)$, is defined with the help of the following integral:

$$L\{f(x); p\} = F(p) = \int_0^\infty e^{-px} f(x)\, dx \tag{8.8}$$

provided that the integral exists. It is to be recollected that the Laplace transform of $f(x)$ exists if the integral in Table 8.1 shows Laplace transform for same elementary functions. Eq. (8.8) is convergent for some designated values of p.

Table 8.1 Laplace Transform for Some Elementary Functions

S.No.	*f(x)*	*L{(x); p}* **or** *F(p)*
1.	1	$1/p,\ p > 0$
2.	x^n, n is positive integer	$n!/p^{n+1},\ p > 0$
3.	$x^n,\ n > -1$	$\Gamma(n+1)/p^{n+1},\ p > 0$
4.	e^{ax}	$1/(p-a),\ p > 0$
5.	$\sin ax$	$a/(p^2+a^2),\ p > 0$
6.	$\cos ax$	$p/(p^2+a^2),\ p > 0$
7.	$\sinh ax$	$a/(p^2-a^2),\ p > \lvert a\rvert$
8.	$\cosh ax$	$p/(p^2-a^2),\ p > \lvert a\rvert$
9.	$J_0(ax)$	$1/\sqrt{p^2+a^2}$
10.	$J_n(ax)$	$\dfrac{\{\sqrt{p^2+a^2}-p\}^n}{a^n\sqrt{p^2+a^2}}$
11.	$\delta(x-a)$	e^{-ap}
12.	$erf\,(\sqrt{x})$	$1/\{p\sqrt{p+1}\}$
13.	x^{-a}	$\Gamma(1-a)p^{a-1}$

8.4 SOME IMPORTANT PROPERTIES OF LAPLACE TRANSFORM

1. Linearity property: If for $i \in \{1,2, ..., n\}$, c_i are constants and $f_i(x)$ are functions with Laplace transforms $F_i(p)$, respectively, then

$$L\{c_1 f_1(x) + \cdots + c_n f_n(x); p\} = c_1 L\{f_1(x); p\} + \cdots + c_n L\{f_n(x); p\}$$

$$L\{c_1 f_1(x) + \cdots + c_n f_n(x); p\} = c_1 F_1(p) + \cdots + c_n F_n(p)$$

* For details of Laplace transform and its inverse, the reader can refer to the recent book on *Integral Transforms* published by RBD.

2. Change of scale property: If $L\{f(x); p\} = F(p)$, then $L\{f(ax); p\} = \frac{1}{a}F(p/a),\ a > 0$

3. First shifting or translation property: If $L\{f(x); p\} = F(p)$, then $L\{e^{-ax}f(x); p\} = F(p + a)$.

4. Second shifting property: If $L\{f(x); p\} = F(p)$ and

$$g(x) = \begin{cases} f(x-a), & x > a \\ 0, & x < a \end{cases},\ \text{then } L\{g(x);\ p\} = e^{-ap}F(p).$$

5. Laplace transform of derivatives: If $L\{f(x); p\} = F(p)$, then

(a) $L\{f'(x);\ p\} = pF(p) - f(0)$,where $f(x)$ is continuous for $0 \le x \le N$ and is of exponential order for $x > N$, while, $f'(x)$ is sectionally continuous for $0 \le x \le N$.

(b) $L\{f''(x);\ p\} = p^2F(p) - pf(0) - f'(0)$, where $f(x)$ and $f'(x)$ are continuous for $0 \le x \le N$ and are of exponential order for $x > N$, while $f''(x)$is sectionally continuous for $0 \le x \le N$.

6. Laplace transform of integrals: If $L\{f(x); p\} = F(p)$, then $L\left\{\int_0^x f(x)\,dx; p\right\} = \frac{F(p)}{p}$

7. Multiplication by powers of x: If $L\{f(x); p\} = F(p)$ then

$$L\{x^n f(x); p\} = (-1)^n \frac{d^n}{dp^n} F(p) = (-1)^n F^n(p)$$

8. Division by x: If $L\{f(x); p\} = F(p)$, then $L\left\{\frac{f(x)}{x}; p\right\} = \int_p^\infty F(u)\,du$, provided $\lim_{x\to 0} \frac{f(x)}{x}$ exists.

9. Initial value theorem: Let $f(x)$ be continuous for all $x \ge 0$ and be of exponential order as $x \to \infty$. Also suppose that $f'(x)$ is of class A. Then,

$$\lim_{x\to 0} f(x) = \lim_{p\to\infty} pL\{f(x); p\}$$

10. Final value theorem: Let $f(x)$ be continuous for all $x \ge 0$ and be of exponential order as $x \to \infty$. Also, suppose that $f'(x)$ is of class A. Then

$$\lim_{x\to\infty} f(x) = \lim_{p\to 0} pL\{f(x); p\}$$

11. Laplace transform of periodic function: Let $f(x)$ be a periodic function with period $\alpha > 0$, i.e., $f(x + n\alpha) = f(x)$, for $n = 1,2,...$

then,
$$L\{f(x); p\} = \frac{1}{1 - e^{-p\alpha}} \int_0^\alpha e^{-px} f(x)\,dx.$$

8.5 INVERSE LAPLACE TRANSFORM

Definition: If the Laplace transform of a function $f(x)$ is $F(p)$, i.e., $L\{f(x); p\} = F(p)$ then $f(x)$ is called *inverse Laplace transform of* $F(p)$, and we express

$$f(x) = L^{-1}\{F(p); x\}$$

where L^{-1} is known as the *inverse Laplace transformation operator*.
Table 8.2 shows inverse Laplace transform of some elementary functions.

Table 8.2 Inverse Laplace Transform* of Some Elementary Functions

S.No.	$F(p)$	$L^{-1}\{F(p); x\}$ or $f(x)$
1.	$1/p$	1
2.	$1/p^{n+1}$ (n is a positive integer)	$x^n/n!$
3.	$1/p^{\alpha+1}$ $\{\mathrm{Re}(\alpha) > -1\}$	$x^\alpha/\Gamma(\alpha+1)$
4.	$1/(p-a)$	e^{ax}
5.	$1/(p^2+a^2)$	$\sin ax/a$
6.	$p/(p^2+a^2)$	$\cos ax$
7.	$1/(p^2-a^2)$	$(\sinh ax)/a$
8.	$p/(p^2-a^2)$	$\cosh ax$
9.	$1/\sqrt{(p^2+a^2)}$	$J_0(ax)$
10.	$1/\{p\sqrt{(p+1)}\}$	$erf(\sqrt{x})$

8.6 SOME IMPORTANT PROPERTIES OF INVERSE LAPLACE TRANSFORM

1. Linearity property: If for all $i \in \{1,2, ..., n\}$, c_i are constants and $F_i(p)$ are the Laplace transforms of $f_i(x)$, respectively, then

$$L^{-1}\{c_1F_1(p) + \cdots + c_nF_n(p); x\} = c_1L^{-1}\{F_1(p); x\} + \cdots + c_nL^{-1}\{F_n(p); x\}$$

$$L^{-1}\{c_1F_1(p) + \cdots + c_nF_n(p); x\} = c_1f_1(x) + \cdots + c_nf_n(x)$$

2. Change of scale property: If $L^{-1}\{F(p); x\} = f(x)$, then

$$L^{-1}\{F(ap); x\} = \frac{1}{a}f(x/a), a > 0\cdot$$

3. First shifting or translation property:

If $L^{-1}\{F(p); x\} = f(x)$, then $L^{-1}\{F(p-a); x\} = e^{ax}f(x) = e^{ax}L^{-1}\{F(p); x\}$.

* The reader is advised to refer to Table 8.1 simultaneously.

4. Second shifting property: If $L^{-1}\{F(p);\ x\} = f(x)$ and

$$g(x) = \begin{Bmatrix} 0 & , & x < a \\ f(x-a) & & x > a \end{Bmatrix},$$

then $L^{-1}\{e^{-ap}F(p);x\} = g(x)$

or $L^{-1}\{e^{-ap}F(p);x\} = f(x-a)\ H(x-a),$

where $H(x - a)$ is the Heaviside unit function.

5. Inverse Laplace transform of derivatives: If $L^{-1}\{F(p);x\} = f(x),$ then

$$L^{-1}\{F^n(p);x\} = L^{-1}\left\{\frac{d^n}{dp^n}F(p);x\right\} = (-1)^n x^n f(x), n = 1, 2, \cdots$$

6. Inverse Laplace transform of integrals: If $L^{-1}\{F(p);\ x\} = f(x)$ then

$$L^{-1}\left\{\int_p^\infty F(u)\ du; x\right\} = \frac{f(x)}{x}.$$

7. Multiplication by powers of *p*: If $L^{-1}\{F(p);\ x\} = f(x)$ and $f(0) = 0$, then $L^{-1}\{pF(p);\ x\} = f'(x)$.

Further, if $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0,$ then the above result is generalised as

$$L^{-1}\{p^n F(p);x\} = f^n(x) = \frac{d^n}{dx^n} f(x).$$

8. Division by powers of *p*: Let $L^{-1}\{F(p);\ x\} = f(x)$. Then,

(a) $L^{-1}\left\{\frac{F(p)}{p};x\right\} = \int_0^x f(u)\,du,$

(b) $L^{-1}\left\{\frac{F(p)}{p^2};x\right\} = \int_0^x\int_0^v f(u)\,du\,dv$

(c) $L^{-1}\left\{\frac{F(p)}{p^n};x\right\} = \int_0^x\int_0^x\cdots\int_0^x f(x)(dx)^n.$

8.7 CONVOLUTION OF TWO FUNCTIONS

The convolution of $f(x)$ and $g(x)$ is expressed and defined as

$$f * g = \int_0^x f(u)g(x-u)\,du$$

or

$$f * g = \int_0^x f(x-u)\,g(u)\,du$$

The convolution theorem (or convolution property)

Let $f(x)$ and $g(x)$ be two functions of class A and let $L^{-1}\{F(p);\, x\} = f(x)$ and $L^{-1}\{G(p);\, x\} = g(x)$. Then,

$$L^{-1}\{F(p)\, G(p);\, x\} = \int_0^x f(u)\, g(x-u)\, du = f * g.$$

One useful form of the above is

$$L\{f * g) = F(p)\, G(p)$$

i.e.,
$$L\left\{\int_0^x f(u)\, g(x-u)\, du\right\} = L\left\{\int_0^x f(x-u)\, g(u)\, du\right\} = F(p)\, G(p).$$

8.8 THE HEAVISIDE EXPANSION FORMULA

Let $F(p)$ and $G(p)$ be polynomials in p, where $F(p)$ has degree less than $G(p)$. Now, if $G(p)$ has n distinct zeros α_r, $r = 1, ..., n$, then

$$L^{-1}\left\{\frac{F(p)}{G(p)};x\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r x}.$$

8.9 THE COMPLEX INVERSION FORMULA

If $f(x)$ has a continuous derivative and is of exponential order γ for large positive values of x, where $\gamma > 0$ and if $F(p) = L\{f(x);\, p\}$, then

$$L^{-1}\{F(p);x\} = f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{px} F(p)\, dp, x > 0$$

and $f(x) = 0;\;\; x < 0$.

8.10 INTEGRAL EQUATIONS IN SPECIAL FORMS

1. Integral equation of convolution type: The integral equation

$$g(x) = f(x) + \int_0^x K(x-t)\, g(t)\, dt$$

wherein kernel $K(x - t)$ is a function of the difference only, is known as *integral equation of convolution type*. Applying the definition of convolution, it can be expressed as

$$g(x) = f(x) + K(x) * g(x)$$

2. Integro-differential equation: An integral equation in which derivatives of the unknown function $g(x)$ are also present is said to be integro-differential equation. For example,

$$\frac{d^2g}{dx^2} = g(x) + \sin x + \int_0^x \cos(x-u)\, g(u)\, du$$

8.11 APPLICATION OF LAPLACE TRANSFORM TO FIND THE SOLUTIONS OF VOLTERRA INTEGRAL EQUATION

8.11.1 Convolution Type Kernels of Volterra Integral Equation: Working Procedure

Let the Volterra integral equation of the first kind be

$$f(x) = \int_0^x K(x-t)\, g(t)\, dt \tag{8.9}$$

or

$$f(x) = K(x) * g(x) \tag{8.10}$$

where kernel $K(x - t)$ is a function of the difference $(x - t)$.

Let $L\{g(x);\, p\} = G(p)$, $\quad L\{K(x);\, p\} = K(p)$

and

$$L\{f(x);\, p\} = F(p) \tag{8.11}$$

Now, applying the Laplace transform to both sides of Eq. (8.10) we get,

$$L\{f(x);\, p\} = L\{K(x)*g(x)\}$$

or $\quad F(p) = K(p)\, G(p)$ (using the convolution theroem)

or

$$G(p) = F(p)/K(p) \tag{8.12}$$

Now, applying the inverse Laplace transform to both sides of Eq. (8.12) we get,

$$g(x) = L^{-1}\left\{\frac{F(p)}{K(p)};x\right\}$$

2. Let Volterra integral equation of the second kind be

$$g(x) = \begin{cases} g(x) = f(x) + \int_0^x K(x-t)\, g(t)\, dt \\ \quad\quad = f(x) + K(x) * g(x) \end{cases} \tag{8.13}$$

Applying the Laplace transform to both sides of Eq. (8.13), we get

$$L\{g(x); p\} = L\{f(x); p\} + L\{K(x) * g(x)\}$$

or $\quad G(p) = F(p) + K(p)G(p)$ [using Eq. (8.11) and the convolution theorem]

or

$$G(p) = \frac{F(p)}{1 - K(p)} \tag{8.14}$$

Now, applying the inverse Laplace transform to both sides of Eq. (8.14), we obtain.

$$g(x) = L^{-1}\left\{\frac{F(p)}{1-K(p)}; x\right\}$$

8.11.2 Resolvent Kernel of Volterra Integral Equation by Using Laplace Transform

In order to find the resolvent kernel of the integral equation [Eq. (8.13)] in which kernel $K(x-t)$ depends on the difference $(x-t)$ by integral transform methods, we first show that if the original kernel $K(x,t)$ is a difference kernel, so is the resolvent kernel.

We know that the resolvent kernel $R(x,t)$ is a sum of its iterated kernels, i.e.,

$$R(x,\ t) = \sum_{m=1}^{\infty} K_m(x,t) = K_1(x,\ t) + K_2(x,t) + \cdots \tag{8.15}$$

[It is to be noted that here, $\lambda = 1$, so, we have used symbol $R(x,t)$ in place of usual symbol $R(x,t;\lambda)$].

Further, the iterated kernels $K_n(x,t)$ are given by

$$K_1(x,t) = K(x,t) \tag{8.16}$$

and $$K_n(x,t) = \int_t^x K(x,z)K_{n-1}(z,t)\,dz, \quad n = 2,3,.... \tag{8.17}$$

Since $K_1(x,t) = K(x,t)$, therefore by Eq. (8.16), we have

$$K_1(x,t) = K(x,t) = K(x-t) \tag{8.18}$$

Taking $n = 2$ in Eq. (8.17), we get

$$K_2(x,t) = \int_t^x K(x,z)\ K_1(z,t)\ dz = \int_t^x K(x-z)K(z-t)\,dz$$

$$K_2(x,t) = \int_0^{x-t} K(x-t-u)K(u)\,du \qquad \text{[By taking } z-t=u\text{]}$$

This shows that $K_2(x,t)$ depends only on the difference $(x-t)$. Proceeding similarly, we can show that $K_3(x,t)$, $K_4(x,t)$ ... also depend only on the difference $(x-t)$. Now, from Eq. (8.15), it follows that the resolvent kernel will also depend only on the difference $(x-t)$. Therefore, we can take that

$$R(x,t) = R(x-t) \tag{8.19}$$

Now, it is known that the solution of Eq. (8.13) is given by

$$g(x) = f(x) + \int_0^x R(x,t)f(t)\,dt$$

or
$$g(x) = f(x) + \int_0^x R(x-t) f(t)\, dt \tag{8.20}$$

Applying Laplace transform to both sides of Eq. (8.20), we get

$$L\{g(x); p\} = L\{f(x); p\} + L\{R(x) * f(x)\}$$

or
$$G(p) = F(p) + R(p) F(p) \tag{8.21}$$

where, $G(p) = L\{g(x); p\}, F(p) = L\{f(x); p\}$ and $R(p) = L\{R(x); p\}$ (8.22)

Using Eq. (8.14) for G(p) in Eq. (8.22), we get

$$\frac{F(p)}{1-K(p)} = F(p)[1 + R(p)]$$

or
$$R(p) = \frac{1}{1-K(p)} - 1 = \frac{K(p)}{1-K(p)} \tag{8.23}$$

Now, applying the inverse Laplace transform to both sides of Eq. (8.23), we get

$$R(x-t) = L^{-1}\left\{\frac{K(p)}{1-K(p)}\right\}$$

8.11.3 Solution of Integral Equations of the Type $f(x) = \int_0^x K(x^2 - t^2) g(t) dt, (x > 0)$ by using Laplace Transform: Working Procedure

Let
$$f(x) = \int_0^x K(x^2 - t^2)\, g(t)\, dt, \qquad (x > 0) \tag{8.24}$$

$$x = u^{1/2}, t = v^{1/2}, g_1(v) = \frac{1}{2} v^{-1/2} g(v^{1/2}), \text{ and } f_1(u) = f(u^{1/2}) \tag{8.25}$$

Then, Eq. (8.24) takes the form

$$f_1(u) = \int_0^u K(u-v)\, g_1(v)\, dv, \qquad (u > 0)$$

$$f_1(u) = K(u) * g_1(u) \tag{8.26}$$

Taking Laplace transform of both sides, we get

$$F_1(p) = K(p).G_1(p) \Rightarrow G_1(p) = \frac{F_1(p)}{K(p)} = \frac{pF_1(p)}{pK(p)} \tag{8.27}$$

Now, let
$$\frac{1}{pK(p)} = H(p). \tag{8.28}$$

Then, Eq. (8.27) becomes

$$G_1(p) = pH(p)\, F_1(p) = L\left\{\frac{d}{du}\int_0^u h(u-v) f_1(v)\, dv\right\}$$

which on taking inverse Laplace transform gives

$$g_1(u) = \frac{d}{du}\int_0^u h(u-v)\; f_1(v)\, dv \tag{8.29}$$

where $h(x) = L^{-1}\{H(p); x\}$.

Finally, from Eq. (8.25) and (8.29), we have the required solution as

$$g(x) = 2\frac{d}{dx}\int_0^x t\, f(t)\, h(x^2 - t^2)\, dt \tag{8.30}$$

EXAMPLE 8.1: Solve the following Abel's integral equation:

(a) $f(x) = \int_0^x \frac{g(t)}{(x-t)^{\alpha}}\, dt, \qquad 0 < \alpha < 1$

(b) $\int_0^x \frac{g(t)}{(x-t)^{1/2}}\, dt = 1 + x + x^2$

Solution: (i) The given integral equation is of convolution type, and therefore expressing it as:

$$f(x) = g(x) * x^{-\alpha} \tag{i}$$

Taking Laplace transform of both sides of Eq. (i) and applying the convolution theorem, we have

$$L\{f(x); p\} = L\{g(x); p\}.L\{x^{-\alpha}; p\}$$

or
$$F(p) = G(p)\frac{\Gamma(1-\alpha)}{p^{1-\alpha}}$$

or
$$G(p) = \frac{p^{1-\alpha} F(P)}{\Gamma(1-\alpha)} = \frac{p}{\Gamma(\alpha)\Gamma(1-\alpha)}\,[\Gamma(\alpha)\, p^{-\alpha} F(p)]$$

$$G(p) = \frac{p}{(\pi / \sin \pi\alpha)}[\Gamma(\alpha)\, p^{-\alpha} F(p)]$$

$$(\because \Gamma(\alpha)\,\Gamma(1-\alpha) = \pi / \sin \pi\alpha)$$

$$G(p) = \frac{p \sin \pi\alpha}{\pi} L\{x^{\alpha-1} * f(x)\},$$

$$G(p) = \frac{\sin \pi\alpha}{\pi} pL\left\{\int_0^x (x-t)^{\alpha-1}\; f(t)\; dt\right\} \quad \text{(By convolution theorem)} \tag{ii}$$

Let $$h(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt \tag{iii}$$

Now, $$L\{h'(x);p\} = pL\{h(x);p\} - h(0) = pL\{(x);p\} \qquad \{\because h(0)=0\}$$

or $$pL\left\{\int_0^x (x-t)^{\alpha-1} f(t)\, dt; p\right\} = L\{h'(x);p\} \qquad \text{[by Eq. (iii)]} \tag{iv}$$

Using Eq. (iv) in Eq. (ii), we get

$$G(p) = \frac{\sin \pi\alpha}{\pi} L\{h'(x);p\}$$

Now, taking inverse, we have

$$g(x) = L^{-1}\{G(p);x\} = \frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\left[\int_0^x (x-t)^{\alpha-1} f(t)\, dt\right]$$

(b) Rewriting the given equation in convolution form, we have

$$g(x) * x^{-1/2} = 1 + x + x^2 \tag{v}$$

Taking Laplace transform of both sides of Eq. (v) and using the convolution theorem,we have

$$L\{g(x)\}\cdot L\{x^{-1/2}\} = L(1) + L\{x\} + L\{x^2\}$$

or $$G(p)\frac{\Gamma(1/2)}{p^{1/2}} = \frac{1}{p} + \frac{1}{p^2} + \frac{2!}{p^3}$$

or $$G(p) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{p^{1/2}} + \frac{1}{p^{3/2}} + \frac{2}{p^{5/2}}\right) \tag{vi}$$

Now, applying the inverse Laplace transform to both sides of Eq. (vi),we obtain

$$g(x) = \frac{1}{\sqrt{\pi}}\left[L^{-1}\left\{\frac{1}{p^{1/2}}\right\} + L^{-1}\left\{\frac{1}{p^{3/2}}\right\} + 2L^{-1}\left\{\frac{1}{p^{5/2}}\right\}\right]$$

$$g(x) = \frac{1}{\sqrt{\pi}}\left[\frac{x^{-1/2}}{\Gamma(1/2)} + \frac{x^{1/2}}{\Gamma(3/2)} + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$g(x) = \frac{1}{\pi}[x^{-1/2} + 2x^{1/2} + (8/3)x^{3/2}], \text{ (Upon simplification)}$$

EXAMPLE 8.2: Solve the integral equation $\sin x = \int_0^x J_0(x-t) g(t)\, dt$.

Solution: The given integral equation may be expressed in convolution form as

$$\sin x = g(x) * J_0(x) \tag{i}$$

Taking Laplace transform of both sides of Eq. (i) and using the convolution theorem, we have

$$L\{\sin x\} = L\{g(x)\}\ L\{J_0(x)\}$$

or
$$\frac{1}{p^2+1} = G(p)\ \frac{1}{\sqrt{p^2+1}}$$

or
$$G(p) = \frac{1}{\sqrt{p^2+1}}$$

Now, taking the inverse Laplace transform, we have

$$g(x) = L^{-1}\{G(p); x\} = J_0(x)$$

EXAMPLE 8.3: Solve the integral equation $x = \int_0^x e^{x-t}\ g(t)\, dt.$

Solution: The integral equation may be written in convolution form as

$$x = e^x * g(x) \tag{1}$$

Now, taking Laplace transform of both sides of Eq. (i) and using the convolution theorem, we get

$$L\{x\} = L\{e^x * g(x)\} = L\{e^x\} \cdot L\{g(x)\}$$

or
$$\frac{1}{p^2} = \frac{1}{p-1} G(p)$$

or
$$G(p) = \frac{p-1}{p^2} = \frac{1}{p} - \frac{1}{p^2} \tag{ii}$$

Taking the inverse Laplace transform of Eq. (ii),we obtain

$$g(x) = L^{-1}\{G(p); x\} = L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{1}{p^2}\right\}$$

or
$$g(x) = 1 - x.$$

EXAMPLE 8.4: Solve the integral equation $g(x) = 1 + \int_0^x \sin(x-t)\ g(t)\ dt$ and verify your answer.

Solution: The given integral equation can be rewritten in convolution form as

$$g(x) = 1 + g(x) * \sin x \tag{i}$$

Taking Laplace transform of both sides of Eq. (i) and using the convolution theorem, we get

$$L\{g(x)\} = L\{1\} + L\{g(x)\}\ L\{\sin x\}$$

or
$$G(p) = \frac{1}{p} + G(p)\frac{1}{p^2+1}$$

or
$$\left(1 - \frac{1}{p^2+1}\right)G(p) = \frac{1}{p}$$

or
$$G(p) = \frac{p^2+1}{p^3} = \frac{1}{p} + \frac{1}{p^3}$$

Applying Laplace inversion, we get

$$g(x) = 1 + \frac{x^2}{2!} = 1 + \frac{x^2}{2} \tag{ii}$$

Verification of solution: Now, we show that Eq. (ii) satisfies the given integral equation.

$$g(x) = 1 + \int_0^x \sin(x-t)\, g(t)\, dt \tag{iii}$$

From Eq. (ii), $g(x) = 1 + (x^2/2)$

Thus, the R.H.S. of Eq. (iii) provides

$$1 + \int_0^x \sin(x-t)(1 + t^2/2)\, dt$$

$$= 1 + \left[\left(1 + \frac{t^2}{2}\right)\cos(x-t)\right]_0^x - \int_0^x t\cos(x-t)\, dt$$

$$= 1 + 1 + \frac{x^2}{2} - \cos x - \left\{[-t\sin(x-t)]_0^x + \int_0^x \sin(x-t)\, dt\right\}$$

$$= 2 + \frac{x^2}{2} - \cos x - [\cos(x-t)]_0^x$$

$$= 2 + \frac{x^2}{2} - \cos x - (1 - \cos x) = 1 + \frac{x^2}{2} = g(x);$$

= L.H.S. of Eq. (iii)

EXAMPLE 8.5: Solve the following Volterra integral equation of the first kind:

$$\int_0^x g(t)g(x-t)dt = 16\sin 4x$$

Solution: The given integral equation can be written as

$$g(x)*g(x) = 16 \sin 4x \tag{i}$$

Taking Laplace transform to both sides of Eq. (i) and applying the convolution theorem, we have

$$L\{g(x)\}^2 = 16\left(\frac{4}{p^2+16}\right)$$

or
$$L\{g(x)\} = \pm\frac{8}{\sqrt{p^2+16}}$$

Now, taking inverse Laplace transform,we have

$$g(x) = L^{-1}\left\{\frac{\pm 8}{\sqrt{p^2+16}}\right\} = \pm 8J_0(4x)$$

EXAMPLE 8.6: Solve the following integral equations:

(a) $g(x) = e^{-x} - 2\int_0^x \cos(x-t)\, g(t)\, dt$

(b) $g(x) = x + \frac{1}{6}\int_0^x (x-t)^3\, g(t)\, dt$

Solution: (a) Rewriting the given integral equation, we have

$$g(x) = e^{-x} - 2g(x)*\cos x \tag{i}$$

Applying Laplace transform to both sides of Eq. (i) and using the convolution theorem, we have

$$L\{g(x)\} = L\{e^{-x}\} - 2L[g(x)]L\{\cos x\}$$

or
$$G(p) = \frac{1}{p+1} - 2G(p)\frac{p}{p^2+1}$$

or
$$G(p) = \frac{p^2+1}{(p+1)^3}$$

or
$$G(p) = \frac{[(p+1)-1]^2+1}{(p+1)^3} \tag{ii}$$

Now, applying inversion of Eq. (ii), we get

$$g(x) = L^{-1}\left[\frac{[(p+1)-1]^2+1}{(p+1)^3}\right]$$

$$g(x) = e^{-x} L^{-1}\left\{\frac{(p-1)^2+1}{p^3}\right\} \qquad \text{(By first shifting theorem)}$$

$$g(x) = e^{-x} L^{-1}\left\{\frac{p^2-2p+2}{p^3}\right\} = e^{-x} L^{-1}\left\{\frac{1}{p} - \frac{2}{p^2} + \frac{2}{p^3}\right\}$$

$$g(x) = e^{-x}\left[L^{-1}\left\{\frac{1}{p}\right\} - 2L^{-1}\left\{\frac{1}{p^2}\right\} + 2L^{-1}\left\{\frac{1}{p^3}\right\}\right]$$

$$g(x) = e^{-x}[1 - 2x + 2(x^2/2!)]$$

$$g(x) = e^{-x}(1 - 2x + x^2)$$

or $$g(x) = e^{-x}(1-x)^2$$

(b) Rewriting the given integral equation, we have

$$g(x) = x + \frac{1}{6}x^3 * g(x) \qquad \text{(iii)}$$

Applying Laplace transform to both sides of Eq. (iii) and using the convolution theorem, we have

$$L\{g(x)\} = L\{x\} + \frac{1}{6}L\{x^3\}.L\{g(x)\}$$

or $$G(p) = \frac{1}{p^2} + \frac{1}{6}\frac{3!}{p^4}G(p)$$

or $$G(p)\left[1 - \frac{1}{p^4}\right] = \frac{1}{p^2}$$

or $$G(p) = \frac{1}{2}\left[\frac{1}{p^2-1} + \frac{1}{p^2+1}\right]$$

Now, applying inversion, we obtain

$$g(x) = \frac{1}{2}(\sinh x + \sin x).$$

EXAMPLE 8.7: Solve the following integro-differential equations

(a) $g'(x) = \sin x + \int_0^x g(t)\cos t\, dt$, where g(0) = 0.

(b) $g'(x) + 5\int_0^x \cos 2(x-t)g(t)dt = 10$, where g(0) = 2.

Solution: (a) The given integral equation can be written as

$$g'(x) = \sin x + g(x)*\cos x \tag{i}$$

Also, given that $g(0) = 0$. (ii)

Applying Laplace transform to both sides of Eq. (i) and using the convolution theorem, we obtain

$$L\{g'(x)\} = L\{\sin x\} + L\{g(x)\}L\{\cos x\}$$

or $$p\,G(p) - g(0) = \frac{1}{p^2+1} + G(p)\frac{p}{p^2+1}$$

or $$\left(1 - \frac{1}{p^2+1}\right)p\,G(p) = \frac{1}{p^2+1} \qquad \text{[Using Eq. (ii)]}$$

or $$G(p) = \frac{1}{p^3} \tag{iii}$$

Now, inverting Eq. (iii) to get

$$g(x) = L^{-1}\left\{\frac{1}{p^3}\right\} = \frac{x^2}{2!} = \frac{x^2}{2}.$$

(b) The given integral equation may be written as

$$g'(x) + 5[\cos 2x * g(x)] = 10 \tag{iv}$$

Also, given that g(0) = 2. (v)

Applying Laplace transform to both sides of Eq. (iv) and using the convolution theorem, we get

$$L\{g'(x)\} + 5L\{\cos 2x\}L\{g(x)\} = 10L\{1\}$$

or $$p\,G(p) - g(0) + 5\frac{p}{p^2+4}G(p) = \frac{10}{p}$$

or $$\left(1 + \frac{5}{p^2+4}\right)pG(p) = \frac{10}{p} + 2, \qquad \text{[Using Eq. (v)]}$$

or $$G(p) = \frac{2p+10}{p^2}\cdot\frac{p^2+4}{p^2+9} = \frac{2p^3+10p^2+8p+40}{p^2(p^2+9)}$$

or $$G(p) = \frac{8}{9p} + \frac{40}{9p^2} + \frac{10p+50}{9(p^2+9)} \tag{vi}$$

Now, taking inverse Laplace transform of Eq. (vi), we have

$$g(x) = L^{-1}\left\{\frac{8}{9p} + \frac{40}{9p^2} + \frac{10p+50}{9(p^2+9)}\right\}$$

$$g(x) = \frac{8}{9}L^{-1}\left\{\frac{1}{p}\right\} + \frac{40}{9}L^{-1}\left\{\frac{1}{p^2}\right\} + \frac{10}{9}L^{-1}\left\{\frac{p}{p^2+9}\right\} + \frac{50}{9}L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$g(x) = \frac{8}{9} + \frac{40}{9}x + \frac{10}{9}\cos 3x + \frac{50}{27}\sin 3x$$

or $$g(x) = \frac{1}{27}(24 + 120x + 30\cos 3x + 50\sin 3x)$$

EXAMPLE 8.8: Find the resolvent kernel of the following Volterra integral equations, and hence, find their solutions:

(a) $g(x) = f(x) + \int_0^x (x-t)\,g(t)\,dt$

(b) $g(x) = f(x) + \int_0^x e^{(x-t)}g(t)\,dt$

Solution: (a) The given integral equation can be written as

$$g(x) = f(x) + g(x)*x \qquad \text{(i)}$$

Applying Laplace transform to both sides of Eq. (i) and using the convolution theorem, we have

$$L\{(g(x)\} = L\{f(x)\} + L\{g(x)\}\cdot L\{x\}$$

or $$G(p) = F(p) + G(p)\frac{1}{p^2}$$

or $$\left(1 - \frac{1}{p^2}\right)G(p) = F(p)$$

or $$G(p) = \frac{p^2}{p^2-1}F(p) \qquad \text{(ii)}$$

Let $R(x-t)$ be the resolvent kernel of the given integral equation. The required solution is given by

$$g(x) = f(x) + \int_0^x R(x-t)\,f(t)\;dt \qquad \text{(iii)}$$

or $$g(x) = f(x) + R(x) * f(x) \qquad \text{(iv)}$$

Applying Laplace transform to both sides of Eq. (iv) and using the convolution theorem, we get

$$L\{g(x)\} = L\{f(x)\} + L\{R(x)\}\cdot L\{f(x)\}$$

or $$G(p) = F(p) + R(p)\;F(p) \qquad [\text{where } R(p) = L\{R(x)\}]$$

or $$\frac{p^2}{p^2-1}F(p) = F(p) + R(p)F(p) \qquad [\text{Using Eq. (ii)}]$$

or $$R(p) = \frac{p^2}{p^2-1} - 1 = \frac{1}{p^2-1}$$

Now, inverting, $R(x) = L^{-1}\{R(p)\} = \sinh x$

so that $R(x - t) = \sinh(x - t)$

giving the required resolvent kernel.

Now, substituting the above value of $R(x - t)$, the required solution by Eq. (iii) is

$$g(x) = f(x) + \int_0^x \sinh(x-t)\, f(t)\, dt$$

(b) The given integral equation can be written as

$$g(x) = f(x) + g(x) * e^x \tag{v}$$

Applying Laplace transform to both sides of Eq. (v) and using the convolution theorem, we have

$$L\{g(x)\} = L\{f(x)\} + L\{g(x)\}.L\{e^x\}$$

or $$G(p) = F(p) + G(p) + G(p)\frac{1}{p-1}$$

or $$\left(1 - \frac{1}{p-1}\right)G(p) = F(p)$$

or $$G(p) = \frac{p-1}{p-2}F(p) \tag{vi}$$

Let $R(x - t)$ be the resolvent kernel of the given integral equation. Then, we know that the required solution is given by

$$g(x) = f(x) + \int_0^x R(x-t) f(t)dt \tag{vii}$$

or $$g(x) = f(x) + R(x) * f(x) \tag{viii}$$

Applying Laplace transform to both sides of Eq. (viii) and using the convolution theorem, we get

$$L\{g(x)\} = L\{f(x)\} + L\{R(x)\}.L\{f(x)\}$$

or $$G(p) = F(p) + R(p)F(p), \text{ where } R(p) = L\{R(x)\}$$

or $$\frac{p-1}{p-2}F(p) = F(p)[1 + R(p)],$$ [Using Eq. (vi) for $G(p)$]

or $$R(p) = \frac{p-1}{p-2} - 1 = \frac{1}{p-2}$$

Now inverting, $R(x) = L^{-1}\left\{\frac{1}{p-2}\right\} = e^{2x}$

Hence, $$R(x-t) = e^{2(x-t)}$$

It is the required resolvent kernel.

And now, substituting the value of $R(x-t)$, the required solution by Eq. (vii) is

$$g(x) = f(x) + \int_0^x e^{2(x-t)} f(t)\,dt$$

EXAMPLE 8.9: Solve the following integral equation

$$f(x) = \int_0^x \frac{g(t)}{(x^2-t^2)^\alpha}\,dt \qquad (0 < \alpha < 1)$$

Solution: Substituting [refer to Eqs. (8.24) and (8.25)]

$$x = u^{1/2},\ t = v^{1/2}, g_1(v) = \frac{1}{2}v^{-1/2}g(v^{1/2}),\ \text{and}\ f_1(u) = f(u^{1/2}) \qquad \text{(i)}$$

Then, the given integral equation takes the form

$$f_1(u) = \int_0^u \frac{g_1(u)}{(u-v)^\alpha}\,dv \qquad \text{(ii)}$$

Now, taking Laplace transform of Eq. (ii), we get

$$F_1(p) = G_1(p)\Gamma(1-\alpha)\,p^{\alpha-1} \quad [\text{since } L\{u^{-\alpha};p\} = \Gamma(1-\alpha)\,p^{\alpha-1}]$$

Thus, $$G_1(p) = \frac{pF_1(p)}{p^\alpha\Gamma(1-\alpha)}$$

and $$H(p) = \frac{1}{p^\alpha\Gamma(1-\alpha)} \qquad \text{[Refer to Eqs. (8.27) and (8.28)]}$$

Then, $$h(x) = L^{-1}\{H(p);x\}$$

$$h(x) = L^{-1}\left\{\frac{1}{p^\alpha\Gamma(1-\alpha)};x\right\} = \frac{\sin\alpha\pi}{\pi}x^{\alpha-1}$$

Thus, solution of the given integral equation by (i) is

$$g(x) = \frac{2\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{t\,f(t)}{(x^2-t^2)^{1-\alpha}}.dt$$

8.12 FOURIER TRANSFORMS AND THEIR IMPORTANT PROPERTIES*

1. The Fourier or complex Fourier transform: Let $f(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuously differentiable and absolutely integrable in $(-\infty, \infty)$. Then, the Fourier transform of $f(x)$, denoted by $F\{f(x); p\}$ or $F(p)$, is defined as

$$F(\{f(x); p\} = F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x) dx, \quad (-\infty, < p < \infty) \tag{8.31}$$

Here, $\frac{1}{\sqrt{2\pi}} e^{-ipx}$ is known as the *kernel of the Fourier transformation* and F represents Fourier transformation operator.

Then, the function $f(x)$ is called *inverse Fourier transform* of $F(p)$ and is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} F(p)\, dp \tag{8.32}$$

2. The (infinite) Fourier sine transform: Let $f(x)$ be a function defined for $x > 0$ and be piecewise continuously differentiable and absolutely integrable in $(0, \infty)$. Then, the Fourier sine transform of $f(x)$, denoted by $F_s\{f(x); p\}$or $F_s(p)$, is defined as

$$F_s\{f(x); p\} = F_s(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(px)\, dx, (p > 0) \tag{8.33}$$

Here, $\sqrt{\frac{2}{\pi}}$ sin px is known as the *kernel of the Fourier sine transform* and F_s represents the Fourier sine transformation operation.

The corresponding inversion formula is then given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(p) \sin(px)\, dp \tag{8.34}$$

3. The (infinite) Fourier cosine transform: Let $f(x)$ be a function defined for $x > 0$ and be piecewise continuously differentiable and absolutely integrable in $(0, \infty)$. Then, the Fourier cosine transform of $f(x)$, denoted by $F_c\{f(x); p\}$ or $F_c(p)$, is defined as

$$F_c\{f(x); p\} = F_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(px)\, dx, \quad (p > 0) \tag{8.35}$$

Here, $\sqrt{\frac{2}{x}} \cos(px)$ is the *kernel of the Fourier cosine transform* and F_c represents the Fourier cosine transformation operation.

* For more details, readers can consult *Integral Transforms* by RBD.

The corresponding inversion formula is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(p) \cos(px)\, dp \tag{8.36}$$

Remarks: (a) If the kernel of the Fourier transform is taken as e^{ipx}, then Eqs. (8.31) and (8.32) take the following forms:

$$F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx \quad \text{and} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} F(p) dp$$

(b) If the kernel of the Fourier sine transform is taken as sin px, then Eqs. (8.33) and (8.34) take the forms as:

$$F_s(p) = \int_0^\infty f(x) \sin\,(px)\; dx \text{ and } f(x) = \frac{2}{\pi} \int_0^\infty F_s(p) \sin px\, dp$$

(c) If the kernel of the Fourier cosine transform is taken as cos px, then Eqs. (8.35) and (8.36) take the forms as

$$F_c(p) = \int_0^\infty f(x) \cos(px)\; dx \text{ and } f(x) = \frac{2}{\pi} \int_0^\infty F_c(p) \cos(px)\, dp$$

4. Linearity of Fourier transforms: Let c_i be constants and $f_i(x)$ [for all $i \in \{1,2,..., n\}$] be functions with Fourier transforms $F_i(p)$, respectively, then

$$\begin{aligned} F\{c_1 f_1(x) + \cdots + c_n f_n(x); p\} &= c_1 F\{f_1(x); p\} + \cdots + c_n F\{f_n(x); p\} \\ &= c_1 F_1(p) + \cdots + c_n F_n(p) \end{aligned}$$

This property holds good for sine and cosine transforms also.

5. Change of scale property:

(a) If $F\{f(x); p\} = F(p)$, then $F\{f(ax); p\} = \frac{1}{a} F(p/a)$.

(b) If $F_s\{f(x); p\} = F_s(p)$, then $F_s\{f(ax); p\} = \frac{1}{a} F_s(p/a)$.

(c) If $F_c\{f(x); p\} = F_c(p)$, then $F_c\{f(ax); p\} = \frac{1}{a} F_c(p/a)$.

6. Shifting property:

If $F\{f(x); p\} = F(p)$, then $F\{f(x-a); p\} = e^{ipa} F(p)$.

7. Convolution or faltung of two functions: The convolution of two integrable functions $f(x)$ and $g(x)$, where $-\infty, < x < \infty$, is expressed and defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) g(x-u)\, du$$

8. The convolution theorem for Fourier transforms–Statement: Let

(a) $f(x)$ and $g(x)$ and their first order derivatives be continuous on (∞, ∞),

(b) $f(x)$ and $g(x)$ be absolutely integrable on (∞, ∞),

(c) $F(p)$ and $G(p)$ be Fouier transforms of $f(x)$ and $g(x)$; respectively. Then, the Fourier transform of the convolution of $f(x)$ and $g(x)$ exists and is the product of the Fourier transforms of $f(x)$ and $g(x)$, i.e.,

$$F\{f * g\} = F\{f(x); p\} \,.F\{g(x); p\}$$

8.13 APPLICATION OF FOURIER TRANSFORM TO DETERMINE THE SOLUTION OF SINGULAR INTEGRAL EQUATIONS

The procedure will be made clear through the following examples:

EXAMPLE 8.10: Solve the integral equation for $f(x)$

$$\int_0^\infty f(x)\cos px\,dx = \begin{cases} 1-p \;, & 0 \le p \le 1 \\ 0 \;, & p > 1 \end{cases}$$

Hence, deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}.$

Solution: Let $\displaystyle\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx = F_c\{f(x)\} = F_c(p).$

Then,
$$F_c(p) = \begin{cases} \sqrt{\dfrac{2}{\pi}}(1-p), & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

Hence, by the Fourier cosine inversion formula,we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(p)\cos px\,dp = \sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}(1-p)\cos px\,dp$$

$$f(x) = \frac{2(1-\cos x)}{\pi x^2}$$

This is the required solution.

Deduction: Substituting the value of $f(x)$ in the given integral equation, we get

$$\int_0^\infty \frac{2(1-\cos x)}{\pi x^2}\cos px\,dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases} \tag{i}$$

Taking $p \to 0$, Eq. (i) yields

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\,dx = 1 \quad \text{or} \quad \int_0^\infty \frac{2\sin^2(x/2)}{x^2}\,dx = \frac{\pi}{2}$$

Putting $x = 2t$, $dx = 2dt$, we get

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}.$$

EXAMPLES 8.11: Solve the following integral equation for $f(x)$:

$$\int_0^\infty f(x)\sin px\,dx = \begin{cases} 1 \ , \ 0 \le p \le 1 \\ 2 \ , \ 1 \le p < 2 \\ 0 \ , \ \ p > 2 \end{cases}$$

Solution: Let $\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty f(x)\sin px\,dx = F_s\{f(x)\} = F_s(p).$

Then, $$F_s(p) = \sqrt{\frac{2}{\pi}}\begin{cases} 1 \ , \ 0 \le p \le 1 \\ 2 \ , \ 1 \le p < 2 \\ 1 \ , \ \ p > 2 \end{cases}$$

Hence, by the Fourier sine inversion formula, we get

$$f(x) = \sqrt{\frac{2}{x}}\int_0^\infty F_s(p)\sin px\,dp$$

$$f(x) = \frac{2}{\pi}\int_0^1 1.\sin px\,dp + \frac{2}{\pi}\int_1^2 2\sin px\,dp + \frac{2}{\pi}\int_2^\infty 0\sin px\,dp$$

$$f(x) = \frac{2}{\pi}\left[\frac{-\cos px}{x}\right]_0^1 + \frac{4}{\pi}\left[\frac{-\cos px}{x}\right]_1^2$$

$$f(x) = \frac{2}{\pi x}[-\cos x + 1] + \frac{4}{\pi x}\{-\cos 2x + \cos x\}$$

$$f(x) = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x)$$

EXERCISE 8.1

1. Show that the solution of the integral equation

$$\int_0^x \frac{g(t)\,dt}{(x-t)^{1/3}} = x(1+x) \quad \text{is} \quad \frac{3\sqrt{3}}{4\pi}x^{(1/3)}(3x+2).$$

2. Solve the following Abel's integral equation

$$\int_0^x \frac{1}{(x-t)^{1/2}} g(t)dt = \sqrt{t}$$

3. Solve the following inhomogeneous integral equations:

(a) $g(x) = x^2 + \int_0^x \sin(x-t)\, g(t)\, dt$

(b) $g(x) = 1 - \int_0^x (x-t)\, g(t)\, dt$

Also, verify the results so obtained.

4. Solve the following integral equations:

(a) $g(x) = c\sin x - 2\int_0^x \cos(x-t)\, g(t)\, dt$

(b) $g(x) = x + 2\int_0^x \cos(x-t)\, g(t)\, dt$

(c) $\int_0^x g(t)\, g(x-t)\, dt = 2g(x) + x - 2$

(d) $\int_0^x g(t)\, g(x-t)\, dt = x + 2g(x)$

5. Show that the only solution of the integral equation

$$\int_0^x g(t)\sin(x-t)\, dt = g(x)$$

is the trivial solution $g(x) = 0$.

6. Solve the following equations:

(a) $\int_0^x g(t)\cos(x-t)\, dt = g'(x) \quad \text{if } g(0) = 1$

(b) $\int_0^x g'(t)\, g(x-t)\, dt = 24x^3 \quad \text{if } g(0) = 0$

(c) $\int_0^x g''(t) g'(x-t)\, dt = g'(x) - g(x) \quad \text{if } g(0) = g'(0) = 0$

(d) $g'(x) = x + \int_0^x g(x-t)\cos t\, dt, \text{ if } g(0) = 4$

7. Solve the following integral equation:

$$\int_0^x f(x)\cos px\, dx = e^{-p}$$

8. Find resolvent kernel of the following integral equation:

$$g(x) = f(x) + \lambda \int_0^x J_0(x-t)\, g(t)\, dt$$

9. Solve the following inhomogeneous Abel's integral equation:

$$g(x) = f(x) + \lambda \int_0^x \frac{g(t)}{(x-t)^{\alpha}}\, dt \qquad (0 < \alpha < 1)$$

10. Solve the integro-differential equation

$$g''(x) + \int_0^x e^{2(x-t)} g'(t) dt = e^{2x}$$

if

$$g(0) = g'(0) = 0$$

Answers

2. $g(t) = \dfrac{1}{2}$

3. (a) $g(x) = x^2 + \dfrac{x^2}{12}$ (b) $g(x) = \cos x$

4. (a) $g(x) = cxe^{-x}$

   (b) $g(x) = x + 2 + 2(x-1)e^x$

   (c) $g(x) = 1$

   (d) $g(x) = J_1(x) - \int_0^x J_0(t)dt$ or $g(x) = 2\delta(x) - J_1(x) + \int_0^x J_0(t)\, dt$

6. (a) $g(x) = 1 + \dfrac{x^2}{2}$ (b) $g(x) = \pm\dfrac{16x^{3/2}}{\sqrt{\pi}}$

   (c) $g(x) = 0$,

   (d) $g(x) = 4 + \dfrac{5}{2}x^2 + \dfrac{1}{24}x^4$

7. $f(x) = \dfrac{2}{\pi(1+x^2)}$

8. $$\frac{\lambda}{\sqrt{(1-\lambda)^2}}\int_0^x\left(\sin\sqrt{1-\lambda^2}\right)(x-t)\frac{J_1(t)}{t}dt+\lambda\left(\cos x\sqrt{1-\lambda^2}\right)$$
$$+\frac{\lambda^2}{\sqrt{1-\lambda^2}}\sin\left(x\sqrt{(1-\lambda^2}\right)$$

9. $$g(x)=f(x)+\int_0^x R(x,t)\,f(t)\,dt$$

where, $R(x,t)=\dfrac{1}{(x-t)\Gamma[n(1-\alpha)]}\displaystyle\sum_{n=1}^{\infty}[\lambda\Gamma(1-\alpha)(x-t)^{1-\alpha}]^n$

10. $g(x)=xe^x-e^x+1$

❖❖❖

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D.C. Sharma • M.C. Goyal

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